COVERINGS OF DIRECTED GRAPHS AND CROSSED PRODUCTS OF C -ALGEBRAS BY COACTIONS OF HOMOGENEOUS SPACES

KLAUS DEICKE, DAVID PASK, AND IAIN RAEBURN

1. INTRODUCTION

The Cuntz-Krieger algebra $C^*(E)$ of a directed graph E is generated by a family of partial isometries satisfying relations which reflect the path structure of E. These *graph algebras* have a rich structure which is determined by the distribution of loops in the graph. Graph algebras have now arisen in many different situations, and there is increasing interest in their interaction with other graph-theoretic ideas.

Here we consider coverings of directed graphs: morphisms $p : F \to E$ of directed graphs which are local isomorphisms. We show that the graph algebra $C^*(F)$ can be recovered from $C^*(E)$ as a crossed product by a coaction of a homogeneous space associated to the fundamental group $\pi_1(E)$. The crossed products which arise this way are unusually tractable because we know so much about graph algebras, and in the second half of the paper we give some evidence that these crossed products of graph algebras will be good models for the general theory of crossed products by homogeneous spaces.

Our results are motivated by work of Kumjian and Pask on *regular coverings*, which are the orbit maps $p : F \to E$ associated to free actions of a group G on a directed graph F [11]. They used a description of F as a skew product $E \times_c G$ due to Gross and Tucker [9, Theorem 2.2.2] to prove that $C^*(F) \times G$ is stably isomorphic to $C^*(E)$. In [10], it was shown that the description of F as a skew product $E \times_c G$ gives a realisation of $C^*(F)$ as a crossed product by a coaction δ_c of G on $C^*(E)$, and the Kumjian-Pask theorem then follows from Katayama duality.

A non-regular covering $p : F \to E$ can also be realised as a kind of skew product, though the fibre is now the homogeneous space $\pi_1(E)/p_*\pi_1(F)$ rather than a group (see [8], [15] or §2 below), and we use this to obtain our description of $C^*(F)$ as a crossed product (Theorem 3.2). It is not in general clear what one should mean by a coaction of a homogeneous space G/H on a C^*-algebra A, but here we actually have a normal coaction δ of the larger group G, and we can use the definition

$$(1.1) \qquad A \times_\delta (G/H) := \overline{\operatorname{span}}\{j_A(a)j_G(f) : a \in A, f \in c_0(G/H)\} \subset M(A \times_\delta G),$$

where (j_A, j_G) is the universal covariant representation of $(A, c_0(G))$ in $M(A \times_\delta G)$. For non-normal coactions, (1.1) is the analogue of a reduced crossed product rather

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than a full one (see Remark 4.1), but for those interested in graph algebras rather than nonabelian duality this definition will suffice.

We begin in Section 2 by reviewing the structure of covering graphs, and give a short proof of a recent result of Pask and Rho [15] which identifies an arbitrary connected covering as a skew product by a homogeneous space. We prove our main theorems in Section 3, and combine them with results from [11] to obtain two corollaries about crossed products of graph algebras, both of which suggest interesting conjectures about crossed products of C^*-algebras by coactions of a discrete group. In the last two sections we prove these conjectures. In Theorem 4.2, we prove that if δ is a coaction of G on A and H is a subgroup of G, then $A \times_\delta G$ decomposes as an iterated crossed product $(A \times (G/H)) \times H$; the surprise is that we can do this with a crossed product rather than a twisted crossed product. In Theorem 5.1, we show that the dual action of H on $A \times_\delta G$ is proper in the sense of Rieffel [22], and that Rieffel's generalized fixed-point algebra can be identified with $A \times_\delta (G/H)$. This is partial confirmation of the observation in [7] that Rieffel's theory of proper actions, as developed in [23, 13, 6], might provide a useful framework for studying crossed products of C^*-algebras by homogeneous spaces.

2. Coverings of directed grpahs

Let $E = (E^0, E^1, r, s)$ be a directed graph. By a *walk* in E we mean a path a in the underlying undirected graph: thus $a = a_1 a_2 \cdots a_n$, where each a_i is either an edge in E or a reverse edge e^{-1} obtained by traversing an edge e backwards. We write $r(e^{-1}) = s(e)$, $s(e^{-1}) = r(e)$, $r(a) = r(a_n)$ and $s(a) = s(a_1)$. The *reduction* of a walk a is obtained by deleting any subwalks of the form ee^{-1} or $e^{-1}e$; the *reduced product* ab of two walks with $r(a) = s(b)$ is the reduction of the concatenation of a and b. With this operation, the reduced walks a with $r(a) = s(a) = u$ form a group $\pi_1(E, u)$; the inverse of a reduced walk a is the walk a^{-1} obtained by traversing a backwards. If E is connected in the sense that there is a walk between any two vertices, then $\pi_1(E, u)$ is up to isomorphism independent of u, and is called the *fundamental group of* E. Thus $\pi_1(E, u)$ by definition consists of reduced loops based at u. This definition of π_1 coincides with that in [24, 2.1.6], for example, because reducing and path equivalence amount to the same thing.

A surjective morphism $p : F \to E$ of directed graphs is a *covering* if for each $v \in F^0$, p maps $r^{-1}(v)$ bijectively onto $r^{-1}(p(v))$ and $s^{-1}(v)$ bijectively onto $s^{-1}(p(v))$ (cf. [24, 2.2.1]). Every covering p takes reduced walks to reduced walks, respects the reduced product and inverses, and has the *unique walk-lifting property*: if a is a reduced walk with range or source $u \in E^0$, then for every $w \in p^{-1}(u)$ there is a unique reduced walk $\tilde{a}_w$ with range or source w such that $p(\tilde{a}_w) = a$. Thus every connected covering $p : F \to E$ induces an injective homomorphism $p_* : \pi_1(F, v) \to \pi_1(E, p(v))$. The index of $p_*\pi_1(F, v)$ is the number of sheets in the covering:

Lemma 2.1. *Let $p : F \to E$ be a connected graph covering and let $v \in F^0$. For $w \in p^{-1}(p(v))$, choose a reduced walk a from w to v. Then $p(a)$ is a reduced loop at $p(v)$, the coset $p(a)p_*\pi_1(F, v)$ is independent of the choice of a, and the map $\theta : w \mapsto p(a)p_*\pi_1(F, v)$ is a bijection of $p^{-1}(p(v))$ onto $\pi_1(E, p(v))/p_*\pi_1(F, v)$.*

Proof. Because $p(w) = p(v)$, $p(a)$ is a loop at $p(v)$, and it is reduced because p maps reduced walks to reduced walks. If b is any other reduced walk in F from w to v, then $a^{-1}b$ is a reduced loop based at v, so $p(a)^{-1}p(b) = p_*(a^{-1}b) \in p_*\pi_1(F, v)$ and $p(b)p_*\pi_1(F, v) = p(a)p_*\pi_1(F, v)$. Hence θ is well-defined.

Now suppose $w, w' \in p^{-1}(p(v))$ satisfy $\theta(w) = \theta(w')$. Let a and a' be reduced walks in F from w and w' to v, so that $\theta(w) = \theta(w')$ says $p(a)p_*\pi_1(F, v) = p(a')p_*\pi_1(F, v)$. Then there is a reduced walk $d \in \pi_1(F, v)$ such that $p(a') = p(a)p(d) = p(ad)$. Since a' and ad both terminate at v and satisfy $p(a') = p(ad)$, it follows by unique walk lifting that $w = w'$. Thus θ is injective. If $b \in \pi_1(E, p(v))$, let $\tilde{b}_v$ be the lift of b in F with range v; then $w := s(\tilde{b}_v) \in p^{-1}(p(v))$ and $\theta(w) = p(\tilde{b}_v)p_*\pi_1(F, v) = bp_*\pi_1(F, v)$. □

A *labelling* of a graph E by a group G is a function $c : E^1 \to G$. If c is a labelling and H is a subgroup of G, the *relative skew product* $E \times_c (G/H)$ is the graph with $(E \times_c (G/H))^i = E^i \times (G/H)$ for $i = 0, 1$, $r(e, gH) = (r(e), gH)$ and $s(e, gH) = (s(e), c(e)gH)$. When H is the trivial subgroup, this is the skew product $E \times_c G$ used in [10] (as opposed to those used in [11] or [9, §2.3.2]).

The maps $(x, gH) \mapsto x$ from $(E \times_c (G/H))^i$ to E^i are a covering of E, and we are going to prove that every connected covering $p : F \to E$ has this form. To define a suitable labelling we choose a *spanning tree* T for E, which is a *tree* (a connected graph for which there is precisely one reduced walk between any two vertices) with the same vertex set as E; every connected directed graph has such a tree (see [24, 2.1.5]). Now fix a vertex $u \in E^0$. Then for each $w \in E^0$, we let a_w be the unique walk in T from u to w, and define $c = c_{u,T} : E^1 \to \pi_1(E, u)$ by $c(e) = a_{s(e)}ea_{r(e)}^{-1}$.

Proposition 2.2. *Let $p : F \to E$ be a connected covering and $v \in F^0$. Choose a spanning tree T for E, and define $c := c_{p(v),T}$. Then F is isomorphic to the skew product $E \times_c (\pi_1(E, p(v))/p_*\pi_1(F, v))$.*

Proof. For $z \in F^0$, $a_{p(z)}$ is the unique reduced walk in T from $u = p(v)$ to $p(z)$. Let $\tilde{a}_z$ be the unique lifting of $a_{p(z)}$ to a walk in F with $r(\tilde{a}_z) = z$, and define $\tau(z) := s(\tilde{a}_z)$, which belongs to $p^{-1}(p(v))$ because $s(a_{p(z)}) = p(v)$. Now define $\phi : F \to E \times_c (\pi_1(E, p(v))/p_*\pi_1(F, v))$ by

$$\phi^0(z) = (p(z), \theta(\tau(z))) \text{ and } \phi^1(e) = (p(e), \theta(\tau(r(e)))).$$

We shall prove that ϕ is the desired isomorphism.

Since p is a graph morphism, it follows immediately from the definition of the skew product that $r(\phi(e)) = \phi(r(e))$. Now

$$\phi(s(e)) = \big(p(s(e)), \theta(\tau(s(e)))\big) \text{ whereas } s(\phi(e)) = \big(p(s(e)), c(p(e))\theta(\tau(r(e)))\big),$$

so we have to prove that the second coordinates agree. Recall from Lemma 2.1 that $\theta(\tau(s(e))) = p(a)p_*\pi_1(F, v)$ where a is any reduced walk in F from $\tau(s(e))$ to v, and we may choose $a := \tilde{a}_{s(e)} e \tilde{a}_{r(e)}^{-1} d$ where d is a reduced walk from $\tau(r(e))$ to v. Then

$$\begin{aligned}\theta(\tau(s(e))) &= p(\tilde{a}_{s(e)} e \tilde{a}_{r(e)}^{-1} d) p_* \pi_1(F, v) \\ &= a_{p(s(e))} p(e) a_{p(r(e))}^{-1} p(d) p_* \pi_1(F, v) \\ &= c(p(e)) p(d) p_* \pi_1(F, v) \\ &= c(p(e)) \theta(\tau(r(e))).\end{aligned}$$

Thus $s(\phi(e)) = \phi(s(e))$, and ϕ is a graph morphism.

To see that ϕ is injective, suppose that $\phi(e) = \phi(f)$. Then $p(e) = p(f)$, and $\tau(r(e)) = \tau(r(f))$ because θ is injective. Since $p(e) = p(f)$ we have $a_{p(r(e))} = a_{p(r(f))}$, and since $\tau(r(e)) = \tau(r(f))$ the liftings $\tilde{a}_{r(e)}$ and $\tilde{a}_{r(f)}$ start at the same vertex; thus we can deduce from unique walk-lifting that $\tilde{a}_{r(e)} = \tilde{a}_{r(f)}$ and $r(e) = r(\tilde{a}_{r(e)}) = r(\tilde{a}_{r(f)}) = r(f)$. But now $p(e) = p(f)$ implies $e = f$ by unique walk-lifting. A similar argument shows that ϕ is injective on F^0.

Now suppose that $(e, bp_* \pi_1(F, v))$ is an edge in $E \times_c (\pi_1(E, p(v))/p_* \pi_1(F, v))$. Let $\tilde{b}_v$ be the lift of b with range v, let d be the lifting of $a_{p(r(e))}$ with $s(d) = s(\tilde{b}_v)$, let $z = r(d)$, and choose $\tilde{e}$ to be the unique lift of e with range z. Then by unique walk-lifting, we have $d = \tilde{a}_z$, so $\tau(r(\tilde{e}))$ is by definition $s(\tilde{a}_z) = s(\tilde{b}_v)$, and when we compute $\theta(\tau(r(\tilde{e})))$ we can take as a the path $\tilde{b}_v$. Thus

$$\phi(\tilde{e}) = \big(p(\tilde{e}), \theta(\tau(r(\tilde{e})))\big) = \big(e, p(\tilde{b}_v) p_* \pi_1(F, v)\big) = \big(e, bp_* \pi_1(F, v)\big),$$

so ϕ is surjective on F^1. A similar argument shows that ϕ is surjective on F^0. □

Remark 2.3. As it stands, the construction of Proposition 2.2 depends on various choices, but none of these really matter. Because the covering is connected, the fundamental group is up to isomorphism independent of the choice of base point v. A different spanning tree T' gives a different labelling c', but it is cohomologous to c in the sense that there is a function $b : E^0 \to \pi_1(E, p(v))$ such that $b(s(e))c'(e) = c(e)b(r(e))$, and the corresponding skew products are isomorphic.

3. Graph algebras and coactions

Our first main theorem realises the C^*-algebra of a covering as a crossed product by a coaction. We therefore begin by reviewing our conventions regarding graph algebras and coactions.

We consider arbitrary directed graphs $E = (E^0, E^1, r, s)$, and Cuntz-Krieger E-families consisting of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with mutually orthogonal ranges which satisfy $S_e^* S_e = P_{r(e)}$ and $P_v = \sum_{s(e)=v} S_e S_e^*$ whenever $0 < |s^{-1}(v)| < \infty$. The graph algebra $C^*(E)$ is generated by a universal Cuntz-Krieger E-family $\{p_v, s_e\}$; for properties of these algebras, see [21] and [1], for example.

For a discrete group G, we denote by $i : G \to UC^*(G)$ the universal unitary representation which generates $C^*(G)$. The comultiplication $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ is the integrated form of the representation $s \mapsto i(s) \otimes i(s) \in U(C^*(G) \otimes C^*(G))$; in this paper, we use only the spatial tensor product of C^*-algebras. A *coaction* of G on a C^*-algebra A is an injective nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$

such that $(\delta\otimes\mathrm{id})\circ\delta=(\mathrm{id}\otimes\delta_G)\circ\delta$. We denote by w_G the function $i:G\to C^*(G)$ viewed as an element of $c_b(G,C^*(G))\subset M(c_0(G)\otimes C^*(G))$. Then a *covariant representation* of (A,G,δ) consists of nondegenerate representations $\pi:A\to B(\mathcal{H})$, $\mu:c_0(G)\to B(\mathcal{H})$ such that

$$(\pi\otimes\mathrm{id})\circ\delta(a)=\mu\otimes\mathrm{id}(w_G)(\pi(a)\otimes 1)\mu\otimes\mathrm{id}(w_G)^*\ \text{ in } M(\mathcal{K}(\mathcal{H})\otimes C^*(G)).$$

The *crossed product* $A\times_\delta G$ is generated by a universal covariant representation (j_A,j_G) in $M(A\times_\delta G)$, and carries a natural dual action $\widehat{\delta}$ of G [20, §2].

Because G is discrete, the *spectral subspaces*

$$A_s:=\{a\in A:\delta(a)=a\otimes i(s)\}$$

together span a dense subspace of A. We write χ_t for the characteristic function $\chi_{\{t\}}$ and a_s for a generic element of A_s; then the elements $j_A(a_s)j_G(\chi_t)$ span a dense subspace of $A\times_\delta G$, and the dual action is characterised by $\widehat{\delta}_r(j_A(a_s)j_G(\chi_t))=j_A(a_s)j_G(\chi_{tr^{-1}})$. These facts are from [18], and the following useful lemma is also implicit there.

Lemma 3.1. *Suppose δ is a coaction of a discrete group G on a C^*-algebra A, and $\pi:A\to B(\mathcal{H})$, $\mu:c_0(G)\to B(\mathcal{H})$ are nondegenerate representations. Then (π,μ) is covariant if and only if*

$$\pi(a_s)\mu(\chi_t)=\mu(\chi_{st})\pi(a_s)\ \text{ for } a_s\in A_s,\ t\in G. \tag{3.1}$$

Proof. Suppose (π,μ) is covariant, so that

$$(\pi(a_s)\otimes i(s))\mu\otimes\mathrm{id}(w_G)=\mu\otimes\mathrm{id}(w_G)(\pi(a_s)\otimes 1). \tag{3.2}$$

Slicing $w_G\in c_b(G,C^*(G))$ by the functional $\chi_{st}:z\mapsto z(st)$ on $C^*(G)$ gives the function χ_{st} in $c_0(G)$, so applying the slice map $\mathrm{id}\otimes\chi_{st}$ to (3.2) gives (3.1).

Now suppose (3.1) holds. It suffices to prove (3.2) for each element a_s of a spectral subspace, and since μ is nondegenerate, it suffices to prove

$$(\pi(a_s)\otimes i(s))\mu\otimes\mathrm{id}(w_G)(\mu(\chi_t)\otimes 1)=\mu\otimes\mathrm{id}(w_G)(\pi(a_s)\otimes 1)(\mu(\chi_t)\otimes 1) \tag{3.3}$$

for every $t\in G$. Using (3.1), the right-hand side of (3.3) reduces to

$$\begin{aligned}\mu\otimes\mathrm{id}(w_G)\big((\pi(a_s)\mu(\chi_t))\otimes 1\big)&=\mu\otimes\mathrm{id}(w_G(\chi_{st}\otimes 1))(\pi(a_s)\otimes 1)\\&=\mu\otimes\mathrm{id}(\chi_{st}\otimes i(st))(\pi(a_s)\otimes 1)\\&=\mu(\chi_{st})\pi(a_s)\otimes i(s)i(t)\\&=(\pi(a_s)\otimes i(s))(\mu(\chi_t)\otimes i(t))\\&=(\pi(a_s)\otimes i(s))\mu\otimes\mathrm{id}(w_G)(\mu(\chi_t)\otimes 1),\end{aligned}$$

and the result follows. □

If H is a non-normal subgroup of G there is some ambiguity about what is meant by the crossed product $A\times_\delta(G/H)$. However, if the coaction δ is normal in the sense that there is a covariant representation (π,μ) of (A,G,δ) with π faithful [17], then the various candidates coincide, and we can define $A\times_\delta(G/H)$ using (1.1). Since the coactions in this section are normal, we defer further discussion of this point till Remark 4.1. With this convention, we can state our main theorem.

Theorem 3.2. *Let $p : F \to E$ be a connected covering of directed graphs and let $v \in F^0$. Then there is a normal coaction δ of $\pi_1(E, p(v))$ on $C^*(E)$ such that*

$$C^*(F) \cong C^*(E) \times_\delta \big(\pi_1(E, p(v))/p_*\pi_1(F, v)\big).$$

To prove this theorem, we use Proposition 2.2 to realise F as the skew product associated to a labelling $c : E^1 \to \pi_1(E, p(v))$. We then recall from [10, Lemma 2.3] that a labelling $c : E^1 \to G$ induces a coaction $\delta_c : C^*(E) \to C^*(E) \otimes C^*(G)$ characterised by

$$\delta_c(s_e) = s_e \otimes i(c(e)) \ \text{ and } \ \delta_c(p_v) = p_v \otimes 1_{C^*(G)}. \tag{3.4}$$

Lemma 3.3. *The coaction δ_c satisfying* (3.4) *is normal.*

Proof. Let γ denote the gauge action of $\mathbb{T}$ on $C^*(E)$, and choose a covariant representation (π, U) of $(C^*(E), \mathbb{T}, \gamma)$ such that π is faithful. Then $\operatorname{Ind}\pi = ((\pi\otimes\lambda)\circ\delta_c, 1\otimes M)$ is a covariant representation of $(C^*(E), G, \delta_c)$, and $\{\pi \otimes \lambda(\delta_c(s_e)), \pi \otimes \lambda(\delta_c(p_v))\}$ is a Cuntz-Krieger E-family in which each projection $\pi \otimes \lambda(\delta_c(p_v)) = \pi(p_v) \otimes 1$ is nonzero. Since

$$((\pi \otimes \lambda) \circ \delta_c)(s_e) = (\pi \otimes \lambda)(s_e \otimes i(c(e))) = \pi(s_e) \otimes \lambda_{c(e)},$$

the representation $U \otimes 1$ implements the gauge action:

$$\begin{aligned} U_z \otimes 1(\pi \otimes \lambda(\delta_c(s_e)))(U_z \otimes 1)^* &= U_z\pi(s_e)U_z^* \otimes \lambda_{c(e)} \\ &= \pi(\gamma_z(s_e)) \otimes \lambda_{c(e)} \\ &= ((\pi \otimes \lambda) \circ \delta_c)(\gamma_z(s_e)). \end{aligned}$$

Thus it follows from the gauge-invariant uniqueness theorem [1, Theorem 2.1] that $(\pi \otimes \lambda) \circ \delta_c$ is faithful. □

Theorem 3.2 will therefore follow from Proposition 2.2 and the next theorem.

Theorem 3.4. *Let $c : E^1 \to G$ be a labelling of the edges of a directed graph E by a group G, and suppose H is a subgroup of G. Then*

$$C^*(E \times_c (G/H)) \cong C^*(E) \times_{\delta_c} (G/H).$$

Proof. The covariance relation (3.1) for (j_A, j_G) extends to give

$$j_A(a_s)j_G(\chi_{tH}) = j_G(\chi_{stH})j_A(a_s) \ \text{ when } a_s \in A_s; \tag{3.5}$$

to see this, just multiply both sides of (3.5) by $j_G(\chi_r)$, and it reduces to (3.1).

We now let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family generating $C^*(E)$, and, following [10, Theorem 2.4], define

$$t_{f,sH} = j_{C^*(E)}(s_f)j_G(\chi_{sH}) \ \text{ and } \ q_{v,tH} = j_{C^*(E)}(p_v)j_G(\chi_{tH}).$$

We claim that $\{t_{f,sH}, q_{v,tH}\}$ is a Cuntz-Krieger $(E \times_c (G/H))$-family. Because the projections p_v lie in the spectral subspace $C^*(E)_{e_G}$ associated to the identity e_G of

G, $j_{C^*(E)}(p_v)$ commutes with $j_G(\chi_{tH})$, and hence the $q_{v,tH}$ are mutually orthogonal projections. Since $s_e \in C^*(E)_{c(e)}$, it follows from (3.5) that

$$\begin{aligned}
t_{e,sH}^* t_{f,tH} &= j_G(\chi_{sH}^*) j_{C^*(E)}(s_e^* s_f) j_G(\chi_{tH}) \\
&= j_{C^*(E)}(s_e^* s_f) j_G(\chi_{c(f)^{-1}c(e)sH}\chi_{tH}) \\
&= \begin{cases} j_{C^*(E)}(s_e^* s_f) j_G(\chi_{tH}) & \text{if } c(f)^{-1}c(e)sH = tH \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} j_{C^*(E)}(p_{r(e)}) j_G(\chi_{tH}) & \text{if } e = f \text{ and } sH = tH \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} q_{r(e),sH} & \text{if } (e, sH) = (f, tH) \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This proves, first, that the $t_{e,sH}$ are partial isometries with initial projections $q_{r(e,sH)}$, and, second, that the $t_{e,sH}$ have mutually orthogonal ranges. A similar calculation shows that

$$t_{e,sH} t_{e,sH}^* = j_{C^*(E)}(s_e s_e^*) j_G(\chi_{c(e)sH}) \leq j_{C^*(E)}(p_{s(e)}) j_G(\chi_{c(e)sH}) = q_{s(e,sH)}.$$

Since $s^{-1}(v, tH) = \{(e, c(e)^{-1}tH) : s(e) = v\}$ is in one-to-one correspondence with $s^{-1}(v)$, we have $0 < |s^{-1}(v, tH)| < \infty$ if and only if $0 < |s^{-1}(v)| < \infty$, and if so,

$$\begin{aligned}
q_{v,tH} = j_{C^*(E)}(p_v) j_G(\chi_{tH}) &= \sum_{s(e)=v} j_{C^*(E)}(s_e s_e^*) j_G(\chi_{tH}) \\
&= \sum_{s(e)=v} j_{C^*(E)}(s_e) j_G(\chi_{c(e)^{-1}tH}) j_{C^*(E)}(s_e)^* \\
&= \sum_{s(e,sH)=(v,tH)} t_{e,sH} t_{e,sH}^*.
\end{aligned}$$

Thus $\{t_{f,sH}, q_{v,tH}\}$ is a Cuntz-Krieger $(E \times_c (G/H))$-family, as claimed. The universal property of the graph algebra now gives us a homomorphism $\pi_{t,q}$ of $C^*(E \times_c (G/H))$ into $C^*(E) \times_{\delta_c} (G/H)$ which takes the canonical generating Cuntz-Krieger family to $\{t_{f,sH}, q_{v,tH}\}$.

The gauge automorphisms γ_z commute with the coaction δ_c in the sense that $\delta_c(\gamma_z(a)) = \gamma_z \otimes \mathrm{id}(\delta_c(a))$ for $a \in C^*(E)$, and hence by the universal property of the crossed product induce automorphisms $\gamma_z \times_{\delta_c} G$ of $C^*(E) \times_{\delta_c} G$. Thus there is a continuous action $\gamma \times_{\delta_c} G$ of $\mathbb{T}$ on $C^*(E) \times_{\delta_c} G$ such that

$$(\gamma \times_{\delta_c} G)_z(j_{C^*(E)}(a) j_G(\chi_t)) = j_{C^*(E)}(\gamma_z(a)) j_G(\chi_t),$$

and which leaves the subalgebra $C^*(E) \times_{\delta_c} (G/H)$ of $M(C^*(E) \times_{\delta_c} G)$ invariant; a calculation shows that $\pi_{t,q} \circ \gamma_z = (\gamma_z \times_{\delta_c} G) \circ \pi_{t,q}$. Since the coaction δ_c is normal, the projections $q_{e,tH}$ are all non-zero, and it follows from the gauge-invariant uniqueness theorem [1, Theorem 2.1] that $\pi_{t,q}$ is injective.

To see that $\pi_{t,q}$ is surjective, it suffices to prove that every $j_{C^*(E)}(s_\mu s_\nu^*) j_G(\chi_{rH})$ belongs to the range of $\pi_{t,q}$, or, equivalently, that every $j_{C^*(E)}(s_\mu) j_G(\chi_{rH}) j_{C^*(E)}(s_\nu^*)$

with $r(\mu) = r(\nu)$ is in the range of $\pi_{t,q}$. But if $\tilde{\mu}$ and $\tilde{\nu}$ are the unique liftings of μ and ν to paths in $E \times_c (G/H)$ with range $(r(\mu), rH)$, then

$$t_{\tilde{\mu}} t_{\tilde{\nu}}^* = j_{C^*(E)}(s_\mu) j_G(\chi_{rH}) j_{C^*(E)}(s_\nu^*),$$

as required. □

Corollary 3.5. *Let δ_c be the coaction of G on $C^*(E)$ induced by a labelling $c : E^1 \to G$, and let H be a subgroup of G. Then $C^*(E) \times_{\delta_c} (G/H)$ is Morita equivalent to $(C^*(E) \times_{\delta_c} G) \times_{\widehat{\delta_c}} H$.*

Proof. The subgroup H acts on the right of $E \times_c G$, and $E \times_c (G/H)$ is isomorphic to the quotient $(E \times_c G)/H$. Thus it follows from Theorem 3.4 that

$$C^*(E) \times_{\delta_c} (G/H) \cong C^*(E \times_c (G/H)) \cong C^*((E \times_c G)/H).$$

From Theorem 1.6 and Corollary 3.1 of [14] we know that $C^*((E \times_c G)/H)$ is Morita equivalent to $C^*(E \times_c G) \times_\beta H$, where β is induced by the right action of H on $E \times_c G$. To finish off, note that the isomorphism of $C^*(E \times_c G)$ onto $C^*(E) \times_{\delta_c} G$ carries β into the restriction of the dual action $\widehat{\delta_c}$. □

Corollary 3.6. *Let δ_c be the coaction of G on $C^*(E)$ induced by a labelling $c : E^1 \to G$, and let H be a subgroup of G. Then there is a coaction δ_d of H on $C^*(E) \times_{\delta_c} (G/H)$ such that*

$$C^*(E) \times_{\delta_c} G \cong (C^*(E) \times_{\delta_c} (G/H)) \times_{\delta_d} H.$$

Proof. Since H acts freely on $E \times_c G$, the Gross-Tucker theorem [9, Theorem 2.2.2] gives a function $d : ((E \times_c G)/H)^1 \to H$ such that $(E \times_c G)/H \times_d H$ is H-isomorphic to $E \times_c G$. Applying Theorem 3.4 with no subgroup (that is, [10, Theorem 2.4]) gives us a coaction δ_d of H such that

$$C^*(E \times_c G) \cong C^*((E \times_c G)/H \times_d H) \cong C^*(E \times_c (G/H)) \times_{\delta_d} H.$$

We finish off using the ismorphisms $C^*(E) \times_{\delta_c} G \cong C^*(E \times_c G)$ of [10, Theorem 2.4] and $C^*(E \times_c (G/H)) \cong C^*(E) \times_{\delta_c} (G/H)$ of Theorem 3.4. □

4. Decomposition of crossed products by discrete coactions

Our next theorem is motivated by Corollary 3.6, which prompts the question: is there always a coaction of H on a crossed product $A \times (G/H)$ such that $A \times G \cong (A \times (G/H)) \times H$? We know from [19, Proposition 6.3] that this is true when G is discrete and H is normal, because the quotient map $G \to G/H$ certainly admits continuous sections when G/H is discrete. Here we show in rather direct fashion that it is still true for non-normal H.

In this section we consider an arbitrary coaction δ of a discrete group G on a C^*-algebra A and a subgroup H of G. We now write $A \times_{\delta,r} (G/H)$ for the C^*-algebra

$$\overline{\operatorname{span}}\{j_A(a) j_G(f) : a \in A, f \in c_0(G/H)\} \subset M(A \times_\delta G),$$

and call it the *reduced crossed product of A by the homogeneous space G/H*.

Remark 4.1. Our reasons for calling this a reduced crossed product are a little delicate. When H is a normal subgroup, it makes sense to restrict a coaction $\delta : A \to A \otimes C^*(G)$ to a coaction $\delta|$ of G/H and form the usual crossed product. If π is a faithful representation of A on $\mathcal{H}$, then the spatially-defined crossed product generated by the operators

$$(\pi \otimes \lambda^{G/H}) \circ \delta|(a), 1 \otimes M(f) \in B(\mathcal{H} \otimes l^2(G/H))$$

has the universal property which characterises $A \times_{\delta|} (G/H)$; since $(\pi \otimes \lambda^{G/H}) \circ \delta|, 1 \otimes M)$ is the regular representation of $(A, G/H, \delta|)$ induced from π, this can be interpreted as saying that the full and reduced crossed products coincide. However, the system $(A, G/H, \delta|)$ also has a natural representation $((\pi \otimes \lambda^G) \circ \delta, 1 \otimes M)$ on $\mathcal{H} \otimes l^2(G)$, where now M is the representation of $c_0(G)$ restricted to $c_0(G/H) \subset M(c_0(G))$, and this representation on $\mathcal{H} \otimes l^2(G)$ makes sense even when H is not normal. So in [3], the reduced crossed product $A \times_{\delta,r} (G/H)$ was by definition the C^*-algebra on $\mathcal{H} \otimes l^2(G)$ generated by the operators $(\pi \otimes \lambda^G) \circ \delta(a)(1 \otimes M(f))$ for $f \in c_0(G/H)$. Since the representation $((\pi \otimes \lambda^G) \circ \delta) \times (1 \otimes M)$ is faithful on $A \times_\delta G$ and everything lies in $M(A \times_\delta G)$, we can view this $A \times_{\delta,r} (G/H)$ as a subalgebra of $M(A \times_\delta G)$. Thus our new definition of $A \times_{\delta,r} (G/H)$ coincides with that of [3].

The notation in [3] was consistent with that of Mansfield [12], who for normal H used $A \times_\delta (G/H)$ to distinguish the subalgebra of $B(\mathcal{H} \otimes l^2(G))$ from the usual crossed product $A \times_{\delta|} (G/H)$ on $\mathcal{H} \otimes l^2(G/H)$. Mansfield proved that when H is normal and amenable $A \times_{\delta|} (G/H)$ is isomorphic to $A \times_{\delta,r} (G/H)$ [12, Proposition 7]. For nonamenable normal subgroups, though, the two algebras need not be isomorphic — for example, if $H = G$ and the coaction is not normal in the sense of Quigg [17].

The Fell-bundle approach gives another way of defining a reduced crossed product. If $\delta : A \to A \otimes C^*(G)$ is a coaction of a discrete group, then the spectral subspaces $\{A_s : s \in G\}$ form a Fell bundle $\mathcal{A}$ over G. When H is a subgroup of G, we can form a Fell bundle $\mathcal{A} \times (G/H)$ over the transformation groupoid $G \times (G/H)$ (see [5, §2]), and this has a regular representation as adjointable operators on a Hilbert module $L^2(\mathcal{A} \times G/H)$ which gives a reduced cross-sectional algebra $C^*_r(\mathcal{A} \times (G/H))$. It is proved in [5, Proposition 2.10] that $C^*_r(\mathcal{A} \times (G/H))$ is naturally isomorphic to the C^*-subalgebra of $M(C^*(\mathcal{A}) \times_{\delta^m} G)$ generated by the images of $C^*(\mathcal{A})$ and $c_0(G/H)$, where δ^m is the maximal coaction on the full cross-sectional algebra $C^*(\mathcal{A})$. However, we know from [4, Lemma 2.1] that there is an isomorphism of $C^*(\mathcal{A}) \times_{\delta^m} G$ onto $A \times_\delta G$ taking $j_{C^*(\mathcal{A})}(C^*(\mathcal{A}))$ to $j_A(A)$ and matching up the copies of $c_0(G/H)$, so our $A \times_{\delta,r} (G/H)$ is isomorphic to $C^*_r(\mathcal{A} \times (G/H))$.

Theorem 4.2. *Let δ be a coaction of a discrete group G on a C^*-algebra A, let H be a subgroup of G, and let $\sigma : G/H \to G$ be a section such that $\sigma(H) = e$. Then there is a normal coaction δ^σ of H on $A \times_{\delta,r} (G/H)$ such that*

$$\delta^\sigma(j_A(a_r)j_G(\chi_{tH})) = j_A(a_r)j_G(\chi_{tH}) \otimes i(\sigma(rtH)^{-1}r\sigma(tH)) \tag{4.1}$$

for $a_r \in A_r$ and $tH \in G/H$, and there is an isomorphism of $(A \times_{\delta,r} (G/H)) \times_{\delta^\sigma} H$ onto $A \times_\delta G$ which carries the dual action $\widehat{\delta^\sigma}$ into the restriction of the dual action $\widehat{\delta}$.

Proof. Define $\phi : G \to H$ by $\phi(s) = \sigma(sH)^{-1}s$, and let ϕ^* denote the homomorphism $f \mapsto f \circ \phi$ of $c_0(H)$ into $c_b(G) = M(c_0(G))$; note that ϕ^* is nondegenerate. The function w_H which takes h to its canonical image $i(h)$ in $UC^*(H)$ is a unitary element of $c_b(H, C^*(H)) \subset M(c_0(H) \otimes C^*(H))$, and is a corepresentation of H. Thus the map $\delta_1 : A \times_\delta G \to (A \times_\delta G) \otimes C^*(H)$ defined by

$$\delta_1(c) := ((j_G \circ \phi^*) \otimes \mathrm{id})(w_H)(c \otimes 1)((j_G \circ \phi^*) \otimes \mathrm{id})(w_H^*) \tag{4.2}$$

is a coaction of H on $A \times_\delta G$. Since δ_1 is nondegenerate as a homomorphism, it extends to $M(A \times_\delta G)$. We claim that this extension of δ_1 maps $A \times_{\delta,r} (G/H)$ into $(A \times_{\delta,r} (G/H)) \otimes C^*(H)$, and therefore restricts to a coaction δ^σ of H on $A \times_{\delta,r} (G/H)$.

For $s \in G$, we have $(\chi_s \otimes 1)(\phi^* \otimes \mathrm{id})(w_H) = \chi_s \otimes i(\phi(s))$. Thus for $a_r \in A_r$ and $tH \in G/H$ we have

$$\begin{aligned}
&(j_G(\chi_s) \otimes 1)\delta_1(j_A(a_r)j_G(\chi_{tH})) \\
&\qquad = \big(j_G(\chi_s)j_A(a_r)j_G(\chi_{tH}) \otimes i(\phi(s))\big)((j_G \circ \phi^*) \otimes \mathrm{id})(w_H^*) \\
&\qquad = \begin{cases} \big(j_A(a_r)j_G(\chi_{r^{-1}s}) \otimes i(\phi(s))\big)((j_G \circ \phi^*) \otimes \mathrm{id})(w_H^*) & \text{if } r^{-1}s \in tH \\ 0 & \text{if } r^{-1}s \notin tH \end{cases} \\
&\qquad = \begin{cases} j_A(a_r)j_G(\chi_{r^{-1}s}) \otimes i(\phi(s)\phi(r^{-1}s)^{-1}) & \text{if } r^{-1}s \in tH \\ 0 & \text{if } r^{-1}s \notin tH \end{cases} \\
&\qquad = (j_G(\chi_s) \otimes 1)\big(j_A(a_r)j_G(\chi_{tH}) \otimes i(\sigma(rtH)^{-1}r\sigma(tH))\big).
\end{aligned}$$

This implies that δ_1 maps $A \times_{\delta,r} (G/H)$ into $(A \times_{\delta,r} (G/H)) \otimes C^*(H)$, and that the restriction $\delta^\sigma := \delta_1|_{A\times(G/H)}$ satisfies (4.1). Since A_e contains an approximate identity for A [18, Corollary 1.6], (4.1) implies that δ^σ is a nondegenerate homomorphism. It follows from the corresponding properties of δ_1 that δ^σ is injective and satisfies the coaction identity. If $\pi \times \mu$ is a faithful representation of $A \times_\delta G$, then $\pi \times \mu$ is faithful on $A \times_{\delta,r} (G/H)$. In the regular representation $\big(((\pi \times \mu) \otimes \lambda^H) \circ \delta^\sigma, 1 \otimes M\big)$ on $\mathcal{H} \otimes l^2(H)$, the algebra component $((\pi \times \mu) \otimes \lambda^H) \circ \delta^\sigma$ is implemented by conjugating the representation $\big((\pi \times \mu)|_{A\times(G/H)}\big) \otimes 1$ by the unitary $((\mu \circ \phi^*) \otimes \lambda^H)(w_H)$, and hence is injective. Thus δ^σ is normal.

To obtain the isomorphism in the Theorem, we prove that if ι is the inclusion of $A \times_{\delta,r} (G/H)$ in $M(A \times_\delta G)$, then $(A \times_\delta G, \iota, j_G \circ \phi^*)$ is a crossed product for $(A \times_{\delta,r} (G/H), H, \delta^\sigma)$, in the sense that $(A \times_\delta G, \iota, j_G \circ \phi^*)$ has properties (a), (b) and (c) of [20, Definition 2.8].

Equation (4.2) implies that $(\iota, j_G \circ \phi^*)$ is covariant for $\delta^\sigma := \delta_1|_{A\times(G/H)}$, so (a) holds. To verify (c), note first that

$$\begin{aligned}
\big(\chi_{tH}\phi^*(\chi_{\phi(t)})\big)(s) = 1 &\iff s \in tH \text{ and } \phi(s) = \phi(t) \\
&\iff sH = tH \text{ and } \sigma(sH)^{-1}s = \sigma(tH)^{-1}t \\
&\iff s = t,
\end{aligned}$$

so that $\chi_{tH}\phi^*(\chi_{\phi(t)}) = \chi_t$. Thus each spanning element $j_A(a_r)j_G(\chi_t)$ of $A \times_\delta G$ can be written as

$$j_A(a_r)j_G(\chi_t) = j_A(a_r)j_G(\chi_{tH})j_G(\phi^*(\chi_{\phi(t)})) = \iota\big(j_A(a_r)j_G(\chi_{tH})\big)j_G \circ \phi^*(\chi_{\phi(t)}),$$

which implies that the elements $\iota(b)j_G \circ \phi^*(\chi_h)$ span a dense subspace of $A \times_\delta G$.

It remains to establish [20, Definition 2.8(b)]. Suppose (ρ,μ) is a covariant representation of $(A\times_{\delta,r}(G/H),H,\delta^\sigma)$. Both $j_A(A)$ and $j_G(c_0(G/H))$ multiply $A\times_{\delta,r}(G/H)$, and hence we can view j_A and j_G as nondegenerate homomorphisms into $M(A\times_{\delta,r}(G/H))$. By [5, Proposition 2.6], we have

$$j_A(a_r)j_G(\chi_{tH})=j_G(\chi_{stH})j_A(a_r)\ \text{ for } a_r\in A_r.$$

Now let $\pi:=\rho\circ j_A$ and $\nu:=\rho\circ j_G|_{c_0(G/H)}$. Then for $a_e\in A_e$, $j_A(a_e)j_G(\chi_{tH})$ belongs to the spectral subspace $(A\times_{\delta,r}(G/H))_e$, and the covariance of (ρ,μ) implies that $\pi(a_e)\nu(\chi_{tH})=\rho(j_A(a_e)j_G(\chi_{tH}))$ commutes with $\mu(\chi_h)$ for $h\in H$. By letting a_e run through an approximate identity in A_e, we conclude that $\nu(\chi_{tH})$ and $\mu(\chi_h)$ commute. Thus they combine to give a representation $\nu\otimes\mu$ of $c_0(G/H)\otimes c_0(H)$. The section σ gives an isomorphism ψ of $c_0(G/H)\otimes c_0(H)$ onto $c_0(G)$ such that $\psi(f\otimes g)(s)=f(sH)g(\phi(s))$; pulling $\nu\otimes\mu$ over gives a representation $\omega:=(\nu\otimes\mu)\circ\psi^{-1}$ of $c_0(G)$ satisfying $\omega(\chi_t)=\nu(\chi_{tH})\mu(\chi_{\sigma(tH)^{-1}t})$ for $t\in G$. We claim that (π,ω) is a covariant representation of (A,G,δ). To see this, let $a_r\in A_r$ and $t\in G$. Then

$$\begin{aligned}\pi(a_r)\omega(\chi_t)&=\rho(j_A(a_r)j_G(\chi_{tH}))\mu(\chi_{\sigma(tH)^{-1}t})\\&=\mu(\chi_{\sigma(rtH)^{-1}r\sigma(tH)\sigma(tH)^{-1}t})\rho(j_A(a_r)j_G(\chi_{tH}))\qquad\text{by (4.1)}\\&=\mu(\chi_{\sigma(rtH)^{-1}rt})\pi(a_r)\nu(\chi_{tH})\\&=\mu(\chi_{\sigma(rtH)^{-1}rt})\nu(\chi_{rtH})\pi(a_r)\\&=\omega(\chi_{rt})\pi(a_r).\end{aligned}$$

It follows from Lemma 3.1 that (π,ω) is covariant, and hence gives a nondegenerate representation $\pi\times\omega$ of $A\times_\delta G$. This extends to $M(A\times_\delta G)$, and satisfies

$$(\pi\times\omega)(j_A(a_r)j_G(\chi_{tH}))=\pi(a_r)\omega(\chi_{tH})=\pi(a_r)\nu(\chi_{tH})=\rho(j_A(a_r)j_G(\chi_{tH}));$$

since $\psi(1\otimes\chi_h)=\phi^*(\chi_h)$, we also have

$$(\pi\times\omega)(j_G\circ\phi^*(\chi_h))=\omega(\phi^*(\chi_h))=\nu\otimes\mu(1\otimes\chi_h)=\mu(\chi_h).$$

This completes the proof that $(A\times_\delta G,\iota,j_G\circ\phi^*)$ is a crossed product for the coaction δ^σ of H on $A\times_{\delta,r}(G/H)$. There is therefore an isomorphism θ of $A\times_\delta G$ onto $(A\times_{\delta,r}(G/H))\times_{\delta^\sigma}H$ such that $\theta\circ\iota=j_{A\times(G/H)}$ and $\theta\circ j_G\circ\phi^*=j_H$. It remains to prove that θ converts $\widehat{\delta}|_H$ into $\widehat{\delta^\sigma}$. But for $k\in H$, both $\widehat{\delta}_k$ and $(\widehat{\delta^\sigma})_k$ fix the copies of A and are given by right translation rt_k on $c_0(H)$ and $c_0(G)$, respectively. Since $\chi_{tH}\phi^*(\chi_h)=\chi_{\sigma(tH)h}$, we have

$$\begin{aligned}(\theta\circ\widehat{\delta}_k)\big(j_A(a_r)j_G(\chi_{tH})j_G(\phi^*(\chi_h))\big)&=\theta\big(\widehat{\delta}_k(j_A(a_r)j_G(\chi_{\sigma(tH)h}))\big)\\&=\theta\big(j_A(a_r)j_G(\chi_{\sigma(tH)hk^{-1}})\big)\\&=\theta\big(j_A(a_r)j_G(\chi_{tH})j_G(\phi^*(\chi_{hk^{-1}}))\big)\\&=j_A(a_r)j_G(\chi_{tH})j_H(\chi_{hk^{-1}})\\&=(\widehat{\delta^\sigma})_k\big(j_A(a_r)j_G(\chi_{tH})j_H(\chi_h)\big)\\&=((\widehat{\delta^\sigma})_k\circ\theta)\big(j_A(a_r)j_G(\chi_{tH})j_G(\phi^*(\chi_h))\big),\end{aligned}$$

as required. □

Corollary 4.3. *Let δ be a coaction of a discrete group G on a C^*-algebra A, and let H be a subgroup of G. Then*

$$(A \times_{\delta,r} (G/H)) \otimes \mathcal{K}(L^2(H)) \cong (A \times_\delta G) \times_{\widehat{\delta}|_H,r} H. \tag{4.3}$$

Proof. It follows from Theorem 4.2 that

$$((A \times_{\delta,r} (G/H)) \times_{\delta^\sigma} H) \times_{\widehat{\delta^\sigma},r} H \cong (A \times_\delta G) \times_{\widehat{\delta}|_H,r} H.$$

Since δ^σ is normal, it follows from [16, Theorem 2.10] that Katayama duality works:

$$((A \times_{\delta,r} (G/H)) \times_{\delta^\sigma} H) \times_{\widehat{\delta^\sigma},r} H \cong (A \times_{\delta,r} (G/H)) \otimes \mathcal{K}(L^2(H)).$$

Composing the isomorphisms gives the result. □

Remarks 4.4. (a) Corollary 4.3 is slightly stronger than [5, Theorem 3.4], which says that $A \times_{\delta,r} (G/H)$ is Morita equivalent to $(A \times_\delta G) \times_{\widehat{\delta}|_H,r} H$. The argument in [5] uses Fell-bundle techniques to build an imprimitivity bimodule which implements the Morita equivalence. We can get a third proof of [5, Theorem 3.4] by combining our Theorem 5.1 below and [22, Corollary 1.7].

(b) There is an analogous Morita equivalence for full crossed products in [5, Theorem 3.1], and it is tempting to ask whether our methods will also give an isomorphism like (4.3) for full crossed products. For the full crossed product $A \times_{\delta,f} (G/H)$ we take the full cross-sectional algebra $C^*(\mathcal{A} \times (G/H))$ of the Fell bundle $\mathcal{A} \times (G/H)$ whose reduced cross-sectional algebra $C^*_r(\mathcal{A} \times (G/H))$ we have already seen to be isomorphic to $A \times_{\delta,r} (G/H)$. We can use the universal property of $C^*(\mathcal{A} \times (G/H))$ to see that there is a coaction $\delta^{\sigma,f}$ of H on $A \times_{\delta,f} (G/H)$ satisfying (4.1), and then the arguments of Theorem 4.2 show that we have

$$\big((A \times_{\delta,f} (G/H)) \times_{\delta^{\sigma,f}} G, H, \widehat{\delta^{\sigma,f}}\big) \cong (A \times_\delta G, H, \widehat{\delta}|_H).$$

If we knew that $\delta^{\sigma,f}$ were *maximal* in the sense of [3], so that

$$(A \times_{\delta,f} (G/H)) \otimes \mathcal{K}(L^2(H)) \cong \big((A \times_{\delta,f} (G/H)) \times_{\delta^\sigma} H\big) \times_{\widehat{\delta^\sigma}} H,$$

then we could deduce a version of Corollary 4.3 for full crossed products. We have, however, been unable to prove that $\delta^{\sigma,f}$ is maximal.

5. The crossed product as a generalized fixed-point algebra

Our last result is suggested by Corollary 3.5 and Corollary 4.3. We expect crossed products by free actions to be related to the fixed-point algebra for the action. Thus these corollaries suggest that we might be able to view $A \times_{\delta,r} (G/H)$ as a fixed-point algebra for the dual action $\widehat{\delta_c}$. We shall confirm this by proving that $\widehat{\delta_c}$ is proper and saturated in the sense of Rieffel [22], so that there is a generalized fixed-point algebra which is Morita equivalent to the reduced crossed product, and that this fixed-point algebra is precisely $A \times_{\delta,r} (G/H)$.

We begin by recalling Rieffel's definition of proper action as it applies to an action α of a discrete group G [22]. He says that α is *proper* if there is a dense α-invariant $*$-subalgebra A_0 of A such that

(i) for every $a, b \in A_0$, the function ${}_E\langle a, b\rangle : s \mapsto a\alpha_s(b^*)$ is in $l^1(G, A)$, and

(ii) for every $a, b \in A_0$, there is a multiplier $\langle a, b\rangle_D$ of A such that $A_0\langle a, b\rangle_D \subset A_0$, $\langle a, b\rangle_D A_0 \subset A_0$, and

$$\sum_{s\in G} c\alpha_s(a^*b) = c\langle a, b\rangle_D \quad \text{for all } c \in A_0. \tag{5.1}$$

(The sum on the left converges absolutely by (i), so (5.1) makes sense; the multipliers $\langle a, b\rangle_D$ satisfying (5.1) are automatically invariant under α.)

If α is proper the *generalized fixed-point algebra* $A^\alpha := \overline{\operatorname{span}}\{\langle a, b\rangle_D : a, b \in A_0\}$ is a C^*-subalgebra of $M(A)$. The action α is *saturated* if $\operatorname{span}\{{}_E\langle a, b\rangle : a, b \in A_0\}$ is dense in the reduced crossed product $A \times_{\alpha,r} G$.

Corollary 1.7 of [22] says that if α is proper and saturated, then $A \times_{\alpha,r} G$ is Morita equivalent to A^α.

Theorem 5.1. *Let δ be a coaction of a group G on a C^*-algebra A, and let H be a subgroup of G. Then the dual action $\widehat{\delta}|_H$ of H on $A \times_\delta G$ is proper and saturated, with generalized fixed-point algebra*

$$(A \times_\delta G)^{\widehat{\delta}|_H} = \overline{\operatorname{span}}\{j_A(a_r)j_G(\chi_{tH}) : r, t \in G, a_r \in A_r\} = A \times_{\delta,r} (G/H). \tag{5.2}$$

Proof. For the dense $\widehat{\delta}|_H$-invariant $*$-subalgebra we take

$$(A \times_\delta G)_0 := \operatorname{span}\{j_A(a_r)j_G(\chi_t) : r, t \in G, a_r \in A_r\},$$

which is dense because $A = \overline{\operatorname{span}}\{a_r\}$ and $c_0(G) = \overline{\operatorname{span}}\{\chi_t\}$, a $*$-subalgebra because $j_A(a_r)j_G(\chi_t) = j_G(\chi_{rt})j_A(a_r)$, and invariant because $\widehat{\delta}_h(j_A(a_r)j_G(\chi_t)) = j_A(a_r)j_G(\chi_{th^{-1}})$.

To verify (i), let $a = j_A(a_r)j_G(\chi_t)$, $b = j_A(b_s)j_G(\chi_u)$, and $h \in H$. Then

$$\begin{aligned} {}_E\langle a, b\rangle(h) &= j_A(a_r)j_G(\chi_t)\widehat{\delta}_h((j_A(b_s)j_G(\chi_u))^*) \\ &= j_A(a_r)j_G(\chi_t)\widehat{\delta}_h(j_G(\chi_u)j_A(b_s)^*) \\ &= j_A(a_r)j_G(\chi_t)j_G(\chi_{uh^{-1}})j_A(b_s)^* \\ &= \begin{cases} j_A(a_r)j_G(\chi_t)j_A(b_s)^* & \text{if } h = t^{-1}u \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{5.3}$$

so the support of $\langle a, b\rangle_E$ consists of the single point $t^{-1}u$. In general $\operatorname{supp}\langle a, b\rangle_E$ is finite for $a, b \in (A \times_\delta G)_0$, so ${}_E\langle a, b\rangle$ is certainly summable. Thus (i) holds.

To show (ii), we define

$$\langle j_A(a_r)j_G(\chi_t), j_A(b_s)j_G(\chi_u)\rangle_D := \begin{cases} j_A(a_r^*b_s)j_G(\chi_{uH}) & \text{if } rt = su \\ 0 & \text{otherwise;} \end{cases} \tag{5.4}$$

once we have shown that these have the required property (5.1), we can find suitable multipliers $\langle a, b\rangle_D$ for general $a, b \in (A \times_\delta G)_0$ by adding up multipliers of the form (5.4). Meanwhile, write $a = j_A(a_r)j_G(\chi_t)$, $b = j_A(b_s)j_G(\chi_u)$ and $c = j_A(c_\ell)j_G(\chi_v)$. Then the formula

$$c\langle a, b\rangle_D = \begin{cases} j_A(c_\ell a_r^* b_s)j_G(\chi_{s^{-1}rv}\chi_{uH}) & \text{if } rt = su \\ 0 & \text{otherwise} \end{cases}$$

and a similar one for $\langle a,b\rangle_D c$ show that $c\langle a,b\rangle_D$ and $\langle a,b\rangle_D c$ belong to $(A\times_\delta G)_0$.

We now verify (5.1) for a, b and c as above. Both sides of (5.1) vanish unless $rt=su$. If $rt=su$ and $h\in H$, then

$$\begin{aligned} c\widehat{\delta}_h(a^*b) &= j_A(c_\ell a_r^* b_s)j_G(\chi_{s^{-1}rv}\chi_{s^{-1}rth^{-1}}) \\ &= \begin{cases} j_A(c_\ell a_r^* b_s)j_G(\chi_{uh^{-1}}) & \text{if } h=v^{-1}t \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

so $\sum_{h\in H} c\widehat{\delta}_h(a^*b)$ is nonzero only if $v^{-1}t\in H$, and then

$$\begin{aligned} \sum_{h\in H} c\widehat{\delta}_h(a^*b) &= j_A(c_\ell a_r^* b_s)j_G(\chi_{ut^{-1}v}) \\ &= j_A(c_\ell a_r^* b_s)j_G(\chi_{s^{-1}rv}) \\ &= j_A(c_\ell a_r^* b_s)j_G(\chi_{s^{-1}rv}\chi_{uH}) \ \text{ because } u^{-1}s^{-1}rv=(v^{-1}t)^{-1}\in H \\ &= c\langle a,b\rangle_D. \end{aligned}$$

Thus $\widehat{\delta}|_H$ is proper.

Since the spectral subspace A_e contains an approximate identity for A [18, Corollary 1.6], taking $s=e$ in (5.3) shows that the elements of the form ${}_E\langle a,b\rangle$ are dense in $\ell^1(H, A\times_\delta G)$, hence in $(A\times_\delta G)\times_{\widehat{\delta},r} H$. Thus $\widehat{\delta}|_H$ is saturated.

It remains to establish (5.2). The formula (5.4) shows that $(A\times_\delta G)^{\widehat{\delta}|_H}$ is contained in the right-hand side. Let $a_r\in A_r$ and $tH\in G/H$. Then if $(u_\lambda)_\lambda$ is an approximate identity for A contained in A_e, we have

$$j_A(a_r)j_G(\chi_{tH}) = \lim_\lambda j_A(u_\lambda a_r)j_G(\chi_{tH}) = \lim_\lambda \langle j_A(u_\lambda)j_G(\chi_{st}), j_A(a_r)j_G(\chi_t)\rangle_D,$$

which is in $(A\times_\delta G)^{\widehat{\delta}|_H}$. $\square$

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Department of Mathematics, University of Paderborn, 33095 Paderborn, Germany

E-mail address: `deicke@uni-paderborn.de`

Mathematics, University of Newcastle, NSW 2308, Australia

E-mail address: `davidp@maths.newcastle.edu.au, iain@maths.newcastle.edu.au`