A DUAL GRAPH CONSTRUCTION FOR HIGHER-RANK GRAPHS, AND K-THEORY FOR FINITE 2-GRAPHS

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ABSTRACT. Given a k-graph Λ and an element p of $\mathbb{N}^k$, we define the dual k-graph, $p\Lambda$. We show that when Λ is row-finite and has no sources, the C^*-algebras $C^*(\Lambda)$ and $C^*(p\Lambda)$ coincide. We use this isomorphism to apply Robertson and Steger's results to calculate the K-theory of $C^*(\Lambda)$ when Λ is finite and strongly connected and satisfies the aperiodicity condition.

1. Introduction

In 1980, Cuntz and Krieger introduced a class of C^*-algebras, now called Cuntz-Krieger algebras, associated to finite $\{0,1\}$-matrices A [4]. Enomoto and Watatani then showed that these algebras could be regarded as being associated in a natural way to finite directed graphs by regarding A as the vertex adjacency matrix of a finite directed graph E [5]. Generalising this association, Enomoto and Watatini associated C^*-algebras $C^*(E)$ to finite graphs E with no sources[1] (E has no sources if each vertex of E is the range of at least one edge). Although not every finite directed graph with no sources has a vertex adjacency matrix with entries in $\{0,1\}$, the vertex adjacency matrix of the dual graph $\widehat{E}$ formed by regarding the edges of E as vertices and the paths of length 2 in E as edges *does* always have entries in $\{0,1\}$, and the Cuntz-Krieger algebras associated to E and to $\widehat{E}$ are canonically isomorphic [11]. These results have since been extended to infinite graphs (see for example [10, 9, 3, 7]; see also [2] when E has sources).

One of the major attractions of graph algebras is their applicability to the classification program for simple purely infinite nuclear C^*-algebras. Conditions on a graph E have been identified which guarantee that $C^*(E)$ is purely infinite, simple, and nuclear, and satisfies the Universal Coefficient Theorem (see, for example, [3]), thus producing a large class of directed graphs whose C^*-algebras are determined up to isomorphism by their K-theory [12]. The K-theory of $C^*(E)$ for an arbitrary directed graph E was calculated in [13], and it is shown in [17] that given any two finitely generated abelian groups G, H such that H is torsion-free, there exists a directed graph E such that $C^*(E)$ is simple, purely infinite, nuclear, and satisfies the Universal Coefficient Theorem, with $K_0(C^*(E)) \cong G$ and $K_1(C^*(E)) \cong H$.

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[1]For the sake of consistency with k-graph notation, we regard directed graphs as 1-graphs, so *no sources* here corresponds to *no sinks* in, for example, [5, 3]

In 1999, Robertson and Steger introduced a class of higher-rank Cuntz-Krieger algebras $\mathcal{A}$, associated to collections $M_1, \ldots, M_k$ of commuting $\{0,1\}$-matrices satisfying appropriate compatibility conditions [15]. In [16], they went on to calculate the K-theory of $\mathcal{A}$, demonstrating in particular that $K_1(A)$ need not be torsion-free, so that the class of higher-rank Cuntz-Krieger algebras exhausts some K-invariants which are not achieved by graph algebras. In order to place these higher-rank Cuntz-Krieger algebras in a graph-theoretic setting, and to generalise them as Watatani and Enomoto had generalised the original Cuntz-Krieger algebras, Kumjian and Pask introduced the notion of a higher-rank graph Λ, and defined and investigated the associated higher-rank graph C^*-algebra $C^*(\Lambda)$ [8]. Connectivity in a rank-k graph Λ is described in terms of k commuting vertex adjacency matrices $\{M_1^\Lambda, \ldots, M_k^\Lambda\}$, called coordinate matrices. Just as in the rank-1 setting, not every k-graph has coordinate matrices with entries in $\{0,1\}$, but if Λ is a k-graph whose coordinate matrices are $\{0,1\}$-matrices, then [8, Corollary 3.5(ii)] shows that $C^*(\Lambda)$ and the C^*-algebra $\mathcal{A}$ associated to the coordinate matrices as in [15] are identical.

In this paper we introduce a notion of a dual graph for higher-rank graphs, and show that for a large class of higher-rank graphs Λ, the dual higher-rank graph $p\Lambda$ and the original higher-rank graph Λ have canonically isomorphic C^*-algebras for all $p \in \mathbb{N}^k$ (c.f. [1]). We also show that by choosing p appropriately, we can ensure that $p\Lambda$ has coordinate matrices with entries in $\{0,1\}$. Using these results, we identify a class of finite rank-2 graphs whose C^*-algebras are isomorphic to the rank-2 Cuntz-Krieger algebras studied by Robertson and Steger, and we use the results of [16] to show that these C^*-algebras are purely infinite, simple, unital and nuclear, and to calculate their K-theory.

The layout of the paper is as follows: in Section 2, we recall the definition of k-graphs and the associated notation; in Section 3, we introduce the dual graph construction for k-graphs, and show that this construction preserves the associated C^*-algebra; and in Section 4, we identify the finite 2-graphs Λ whose C^*-algebras can be studied using Robertson and Steger's results, and use these results to calculate $K_*(C^*(\Lambda))$.

In the final stages of preparation of this paper, the authors became aware of Evans' Ph.D. thesis [6], which appears to obtain more general results regarding K-theory for 2-graph C^*-algebras than those established here. The authors thank the referee for helpful comments which we feel have improved the exposition.

2. Preliminaries

We regard $\mathbb{N}^k$ as an additive semigroup with identity 0. Given $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum and $m \wedge n$ for their coordinate-wise minimum, and if $m \le n$, then we write $[m, n]$ for the set $\{p \in \mathbb{N}^k : m \le p \le n\}$. We denote the canonical generators of $\mathbb{N}^k$ by $\{e_1, \ldots, e_k\}$, and for $n \in \mathbb{N}^k$, we write n_j for the j^{th} coordinate of n).

Definition 2.1. Let $k \in \mathbb{N} \setminus \{0\}$. A *$k$-graph* is a pair (Λ, d) where Λ is a countable category and d is a functor from Λ to $\mathbb{N}^k$ which satisfies the *factorisation property*: if $\lambda \in \mathrm{Mor}(\Lambda)$ and $d(\lambda) = m + n$, then there are unique morphisms $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

We refer to elements of $\mathrm{Mor}(\Lambda)$ as *paths* and to elements of $\mathrm{Obj}(\Lambda)$ as *vertices* and we write r and s for the codomain and domain maps. The factorisation property allows us to identify $\mathrm{Obj}(\Lambda)$ with $\{\lambda \in \mathrm{Mor}(\Lambda) : d(\lambda) = 0\}$. So we write $\lambda \in \Lambda$ in place of $\lambda \in \mathrm{Mor}(\Lambda)$, and when $d(\lambda) = 0$, we regard λ as a vertex of Λ.

Given $\lambda \in \Lambda$ and $E \subset \Lambda$, we define $\lambda E = \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$ and $E\lambda = \{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\}$. In particular if $d(v) = 0$, then v is a vertex of Λ and $vE = \{\lambda \in E : r(\lambda) = v\}$; similarly, $Ev = \{\lambda \in \Lambda : s(\lambda) = v\}$. We write Λ^n for the collection $\{\lambda \in \Lambda : d(\lambda) = n\}$.

Definition 2.2. We say that a k-graph (Λ, d) is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and that Λ has *no sources* if $v\Lambda^n$ is nonempty for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that Λ is *strongly connected* if $v\Lambda w$ is nonempty for all $v, w \in \Lambda^0$, and we say that Λ is *finite* if Λ^0 and each Λ^{e_i} are finite.

The factorisation property ensures that if $l \leq m \leq n \in \mathbb{N}^k$ and if $d(\lambda) = n$, then there exist unique paths denoted $\lambda(0, l)$, $\lambda(l, m)$ and $\lambda(m, n)$ such that $d(\lambda(0, l)) = l$, $d(\lambda(l, m)) = m - l$, and $d(\lambda(m, n)) = n - m$ and such that $\lambda = \lambda(0, l)\lambda(l, m)\lambda(m, n)$.

Given $k \in \mathbb{N} \setminus \{0\}$, and k-graphs (Λ_1, d_1) and (Λ_2, d_2), we call a covariant functor $x : \Lambda_1 \to \Lambda_2$ a *graph morphism* if it satisfies $d_2 \circ x = d_1$.

Definition 2.3. As in [8], given $k \in \mathbb{N} \setminus \{0\}$, we write Ω_k for the k-graph given by $\mathrm{Obj}(\Omega_k) = \mathbb{N}^k$, $\mathrm{Mor}(\Omega_k) = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$, $r(m, n) = m$, $s(m, n) = n$, $(m, n) \circ (n, p) = (m, p)$, and $d(m, n) = n - m$. Given a k-graph Λ, an *infinite path* of Λ is a graph morphism $x : \Omega_k \to \Lambda$. We denote the collection of all infinite paths of Λ by Λ^∞. For $p \in \mathbb{N}^k$, we write $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ for the shift-map determined $\sigma^p(x)(m, n) = x(m + p, n + p)$, and we say that $x \in \Lambda^\infty$ is *aperiodic* if there do not exist $p, q \in \mathbb{N}^k$ with $p \neq q$ and $\sigma^p(x) = \sigma^q(x)$.

Definition 2.4. Let (Λ, d) be a row-finite k-graph with no sources. A Cuntz-Krieger Λ-family is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

(i) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(ii) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(iii) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(iv) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The *Cuntz-Krieger algebra* $C^*(\Lambda)$ is the C^*-algebra generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the sense that for every Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ there is a unique homomorphism π of $C^*(\Lambda)$ satisfying $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

3. Dual Higher Rank Graphs

In this section we define the higher rank analog $p\Lambda$ of the dual graph construction for directed graphs.

Definition 3.1. Let (Λ, d) be a k-graph and let $p \in \mathbb{N}^k$. Let $p\Lambda = \{\lambda \in \Lambda : d(\lambda) \geq p\}$. Define range and source maps on $p\Lambda$ by $r_p(\lambda) = \lambda(0, p)$, and $s_p(\lambda) = \lambda(d(\lambda) - p, d(\lambda))$ for all $\lambda \in p\Lambda$, and define composition by $\lambda \circ_p \mu = \lambda\mu(p, d(\mu)) = \lambda(0, d(\lambda) - p)\mu$ whenever $s_p(\lambda) = r_p(\mu)$. Finally, define a degree map d_p on $p\Lambda$ by $d_p(\lambda) = d(\lambda) - p$ for all $\lambda \in p\Lambda$.

Proposition 3.2. *Let (Λ, d) be a k-graph, and let $p \in \mathbb{N}^k$. Then $(p\Lambda, d_p)$ is a k-graph.*

Proof. It is straightforward to check that $p\Lambda$ is a category with the indicated operations. If $\lambda, \mu \in p\Lambda$ and $s_p(\lambda) = r_p(\mu)$, then $\lambda \circ_p \mu = \lambda\mu(p, d(\mu))$ by definition, so $d_p(\lambda \circ_p \mu) = d(\lambda) + d(\mu) - 2p = d_p(\lambda) + d_p(\mu)$. So d_p is a functor from $p\Lambda$ to $\mathbb{N}^k$.

We need to check that the factorisation property holds for $p\Lambda$. Take any $\lambda \in p\Lambda$ and $m, n \in \mathbb{N}^k$ with $m + n = d_p(\lambda)$, so $d(\lambda) = m + p + n$. By the factorisation property for Λ we have $\lambda = \lambda(0, m)\lambda(m, m + p)\lambda(m + p, m + p + n)$. But then $\lambda = \big(\lambda(0, m)\lambda(m, m + p)\big) \circ_p \big(\lambda(m, m + p)\lambda(m + p, m + p + n)\big)$ in $p\Lambda$, and $d_p(\lambda(0, m)\lambda(m, m + p)) = m$ and $d_p(\lambda(m, m + p)\lambda(m + p, m + p + n)) = n$. This decomposition is unique by the factorisation property for Λ. $\square$

Remark 3.3. If Λ has no sources, then $p\Lambda$ has no sources, and if Λ is row-finite, then $p\Lambda$ is row-finite.

Proposition 3.4. *Let (Λ, d) be a k-graph, and let $p, q \in \mathbb{N}^k$. Then $q(p\Lambda) = (q+p)\Lambda$.*

Proof. By definition, we have $q(p\Lambda)^n = p\Lambda^{(n+q)} = \Lambda^{(n+q+p)} = (q + p)\Lambda^n$ for all $n \in \mathbb{N}$. Hence $q(p\Lambda)$ and $(q + p)\Lambda$ have identical elements. For the remainder of the proof, we write $s^{p\Lambda}_q$, $r^{p\Lambda}_q$, $\circ^{p\Lambda}_q$, and $d^{p\Lambda}_q$ for the source, range, composition and degree maps of the dual graph $q(p\Lambda)$.

Fix $\lambda \in \Lambda^{n+p+q}$. We have $s_{(q+p)}(\lambda) = \lambda(n, n + p + q)$ by definition, while $s^{p\Lambda}_q(\lambda)$ is the final segment μ of λ such that $d(\mu) - p = d_p(\mu) = q$; that is $d(\mu) = p + q$. Hence $s_{p+q}(\lambda) = s^{p\Lambda}_q(\lambda)$. Similarly, $r_{p+q}(\lambda) = \lambda(0, p + q) = r^{p\Lambda}_q(\lambda)$. Moreover, $d_{p+q}(\lambda) = d(\lambda) - (p + q) = d_p(\lambda) - q = d^{p\Lambda}_q(\lambda)$. Since λ was arbitrary, it follows that the range, source, and degree maps for $(p + q)\Lambda$ and $q(p\Lambda)$ agree.

This established, we have $r_{p+q}(\lambda) = s_{p+q}(\mu)$ if and only if $r^{p\Lambda}_q(\lambda) = s^{p\Lambda}_q(\mu)$, in which case both $\lambda \circ_{p+q} \mu$ and $\lambda \circ^{p\Lambda}_q \mu$ are equal to $\lambda\mu(p + q, d(\mu))$ by definition, completing the proof. $\square$

Theorem 3.5. *Let (Λ, d) be a row-finite k-graph with no sources, and let $p \in \mathbb{N}^k$. Let $\{s_\lambda : \lambda \in \Lambda\}$ denote the universal generating Cuntz-Krieger Λ-family in $C^*(\Lambda)$, and let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal generating Cuntz-Krieger $p\Lambda$-family in $C^*(p\Lambda)$. For all $\lambda \in p\Lambda$, define $r_\lambda = s_\lambda s^*_{s_p(\lambda)}$. There is an isomorphism $\phi : C^*(p\Lambda) \to C^*(\Lambda)$ such that $\phi(t_\lambda) = r_\lambda$ for all $\lambda \in p\Lambda$.*

Proof. First we show that the family $\{r_\lambda : \lambda \in p\Lambda\}$ is a Cuntz-Krieger $p\Lambda$-family. Since, for any $\beta \in p\Lambda^0$, we have $s_\beta \neq 0$ it follows that $r_\beta = s_\beta s^*_\beta \neq 0$ and that it is a projection in $C^*(\Lambda)$. Furthermore, for distinct $\alpha, \beta \in p\Lambda^0$, we have

$$r_\alpha r_\beta = s_\alpha s^*_\alpha s_\beta s^*_\beta = \delta_{\alpha,\beta} s_\alpha s^*_\beta = \delta_{\alpha,\beta} r_\alpha.$$

This establishes relation (i).

For relation (ii), let $\mu, \nu \in p\Lambda$ with $r_p(\nu) = s_p(\mu)$, so $\mu \circ_p \nu = \mu\nu(p, d(\nu))$. Then,

$$r_{\mu\circ_p\nu} = s_{\mu\circ_p\nu} s^*_{s_p(\mu\circ_p\nu)} = s_\mu s_{\nu(p,d(\nu))} s^*_{s_p(\nu)} = s_\mu s^*_{s_p(\mu)} s_{s_p(\mu)} s_{\nu(p,d(\nu))} s^*_{s_p(\nu)}. \tag{3.1}$$

But $s_p(\mu) = r_p(\nu) = \nu(0, p)$, so we can rewrite the right-hand side of (3.1) to obtain $r_{\mu\circ_p\nu} = s_\mu s^*_{s_p(\mu)} s_\nu s^*_{s_p(\nu)} = r_\mu r_\nu$. This establishes relation (ii).

Let $\lambda \in p\Lambda$, say $d_p(\lambda) = n$. Then $r^*_\lambda r_\lambda = s_{s_p(\lambda)} s^*_\lambda s_\lambda s^*_{s_p(\lambda)} = s_{s_p(\lambda)} s^*_{s_p(\lambda)} = r_{s_p(\lambda)}$ by definition, establishing relation (iii).

Finally, for relation (iv), let $\beta \in p\Lambda^0$ and let $n \in \mathbb{N}^k$. Then

$$r_\beta = s_\beta s^*_\beta = \sum_{\gamma \in s(\beta)\Lambda^n} s_\beta s_\gamma s^*_\gamma s^*_\beta = \sum_{\lambda \in \beta\Lambda^n} s_\lambda s^*_\lambda.$$

Applying the factorisation property and relation (ii) for $C^*(\Lambda)$ to the right-hand side then gives

$$r_\beta = \sum_{\lambda \in \beta\Lambda^n} s_{\lambda(0,n)} s_{\lambda(n,n+p)} s^*_{\lambda(n,n+p)} s^*_{\lambda(0,n)},$$

and then since each $s_{\lambda(n,n+p)} s^*_{\lambda(n,n+p)}$ is a projection, we obtain

$$r_\beta = \sum_{\lambda \in \beta\Lambda^n} (s_{\lambda(0,n)} s_{\lambda(n,n+p)} s^*_{\lambda(n,n+p)})(s_{\lambda(n,n+p)} s^*_{\lambda(n,n+p)} s^*_{\lambda(0,n)}) = \sum_{\lambda \in \beta(p\Lambda^n)} r_\lambda r^*_\lambda,$$

which establishes relation (iv).

It follows from the universal property of $C^*(p\Lambda)$ that there exists a homomorphism $\phi : C^*(p\Lambda) \to C^*(\Lambda)$ satisfying $\phi(t_\lambda) = r_\lambda$ for all $\lambda \in p\Lambda$. We claim that $\{r_\lambda : \lambda \in p\Lambda\}$ generates $C^*(\Lambda)$. To see this, let $\sigma \in \Lambda$ with $d(\sigma) = n$. An application of relation (iv) for $C^*(\Lambda)$ gives $s_\sigma = \sum_{\beta \in s(\sigma)\Lambda^p} s_\sigma s_\beta s^*_\beta = \sum_{\lambda \in \sigma\Lambda^p} s_\lambda s^*_{s_p(\lambda)}$, and this last is equal to $\sum_{\lambda \in \sigma\Lambda^p} r_\lambda$ by definition. Thus ϕ maps $C^*(p\Lambda)$ onto $C^*(\Lambda)$.

Now let γ^Λ denote the gauge action on $C^*(\Lambda)$, and let $\gamma^{p\Lambda}$ denote the gauge action on $C^*(p\Lambda)$. For $z \in \mathbb{T}^k$ and $\lambda \in p\Lambda$, we have

$$\gamma^\Lambda_z(r_\lambda) = \gamma^\Lambda_z(s_\lambda s^*_{s_p(\lambda)}) = z^{d(\lambda)} s_\lambda (z^{d(s_p(\lambda))} s_{s_p(\lambda)})^* = z^{d(\lambda)-p} r_\lambda = \gamma^{p\Lambda}(r_\lambda).$$

Theorem 3.4 of [8] now establishes that ϕ is injective. $\square$

Remark 3.6. The hypotheses that Λ be row-finite and have no sources are crucial in Theorem 3.5. To see why, notice that for $v \in \Lambda^0$, the generator s_v of $C^*(\Lambda)$ is recovered in $C^*(p\Lambda)$ as $\sum_{\beta \in p\Lambda^0, r(\beta)=v} r_\beta$. However, even for 1-graphs the Cuntz-Krieger relations only insist that $p_v = \sum_{r(e)=v} s_e s^*_e$ when $r^{-1}(v)$ is finite and nonempty.

Lemma 3.7. *Let (Λ, d) be a k-graph, and let $p \in \mathbb{N}^k$. For each $n \in \mathbb{N}^k$ with $n \le p$ and $v, w \in p\Lambda^0$, there is at most one $\lambda \in v(p\Lambda^n)w$.*

Proof. Let $v, w \in p\Lambda^0 = \Lambda^p$ and suppose $\lambda \in v(p\Lambda^n)w$. Then $\lambda \in \Lambda^{n+p}$, $\lambda(0,p) = v$, and $\lambda(n, n+p) = w$. Since $n \le p$ we have $\lambda(0,n) = \big(\lambda(0,p)\big)(0,n) = v(0,n)$, so $\lambda = \lambda(0,n)\lambda(n,n+p) = v(0,n)w$, and hence is determined by v and w. $\square$

Notation 3.8. Let (Λ, d) be a k-graph. We write M^Λ_i, $1 \le i \le k$ for the matrices in $M_{\Lambda^0}(\mathbb{N})$ defined by $(M^\Lambda_i)_{v,w} = |w\Lambda^{e_i}v|$ for $w, v \in \Lambda^0$, and we refer to these matrices as the *coordinate matrices* of Λ.

Remark 3.9. In [8, 6] $(M^\Lambda_i)_{v,w} = |v\Lambda^{e_i}w|$, so our M^Λ_i is the transpose of theirs. This is for consistency with the matrices in [16, 15]; we will be employing Robertson and Steger's results to calculate K-theory in Section 4.

Corollary 3.10. *Let (Λ, d) be a k-graph, and let and $p \in \mathbb{N}^k$ with $p_i \ge 1$ for $1 \le i \le k$. Then the coordinate matrices $M^{p\Lambda}_i$ of $p\Lambda$ are $\{0,1\}$-matrices.*

4. K-theory

In this section we identify a class of 2-graphs whose associated C^*-algebras are isomorphic to higher rank Cuntz-Krieger algebras in the sense of [16], and use the results of [16] to calculate the K-theory of the C^*-algebras of such 2-graphs. To state the main theorem for this section we employ the following notation: given square $n \times n$ matrices M, N, we write $\begin{bmatrix} M & N \end{bmatrix}$ for the block $n \times 2n$ matrix whose first n columns are those of M and whose last n columns are those of N. We also write $\mathbf{1}$ for the element $(1,1)$ of $\mathbb{N}^2$.

Theorem 4.1. *Let (Λ, d) be a 2-graph which is finite and strongly connected as in Definition 2.2 and which has an aperiodic infinite path as in Definition 2.3. Then $C^*(\Lambda)$ is purely infinite, simple, unital and nuclear, and we have*

$$\begin{aligned} \operatorname{rank}(K_0(C^*(\Lambda))) &= \operatorname{rank}(K_1(C^*(\Lambda))) \\ &= \operatorname{rank}\big(\operatorname{coker}\big[I - M_1^{\mathbf{1}\Lambda} \quad I - M_2^{\mathbf{1}\Lambda}\big]\big) \qquad (4.1)\\ &\qquad + \operatorname{rank}\big(\operatorname{coker}\big[I - \big(M_1^{\mathbf{1}\Lambda}\big)^t \quad I - \big(M_2^{\mathbf{1}\Lambda}\big)^t\big]\big); \end{aligned}$$

$$\operatorname{tor}(K_0(C^*(\Lambda))) \cong \operatorname{tor}\big(\operatorname{coker}\big[I - M_1^{\mathbf{1}\Lambda} \quad I - M_2^{\mathbf{1}\Lambda}\big]\big);\ \textit{and} \tag{4.2}$$

$$\operatorname{tor}(K_1(C^*(\Lambda))) \cong \operatorname{tor}\big(\operatorname{coker}\big[I - \big(M_1^{\mathbf{1}\Lambda}\big)^t \quad I - \big(M_2^{\mathbf{1}\Lambda}\big)^t\big]\big). \tag{4.3}$$

The remainder of this section constitutes the proof of Theorem 4.1. We begin by recalling some definitions from [16]. Let A be a finite set, and let M_1, M_2 be $A \times A$ matrices with entries in $\{0, 1\}$. For $n \in \mathbb{N}^k$, let $W_n = \{w : [0, n] \to A : M_j(w(l + e_j), w(l)) = 1$ whenever $l, l + e_j \in [0, n]\}$; we refer to the elements of W_n as *allowable words of shape* n, and write W for the collection $\bigcup_{n\in\mathbb{N}^2} W_n$ of all *allowable words.* For $u \in W$, write $S(u)$ for the shape of u; that is, $S(u)$ is the unique element of $\mathbb{N}^2$ such that $u \in W_{S(u)}$. We identify W_0 with A. The matrices M_1, M_2 are said to satisfy (H0)–(H3) if

- (H0) Each M_i is nonzero;
- (H1a) $M_1M_2 = M_2M_1$;
- (H1b) M_1M_2 is a $\{0, 1\}$-matrix;
- (H2) the directed graph with a vertex for each $a \in A$ and a directed edge (a, i, b) from a to b for each a, i, b such that $M_i(b, a) = 1$ is irreducible; and
- (H3) for each $m \in \mathbb{Z}^2 \setminus \{0\}$, there exists a word $w \in W$ and elements l_1, l_2 of $\mathbb{N}^2$ with $0 \le l_1, l_2 \le S(w)$ such that $l_2 - l_1 = m$ and $w(l_1) \ne w(l_2)$.

Notation 4.2. If (Λ, d) is a 2-graph such that the coordinate matrices M_1^Λ and M_2^Λ are $\{0, 1\}$-matrices, we write W_n^Λ and W^Λ for the collection of allowable words of shape n and for the collection of all allowable words respectively. For $\lambda \in \Lambda$, let w_λ^Λ be the word in $W_{d(\lambda)}^\Lambda$ given by $w_\lambda^\Lambda(m) = s(\lambda(0, m))$ for $0 \le m \le d(\lambda)$. Since each M_i^Λ is a $\{0, 1\}$-matrix, the map $\lambda \mapsto w_\lambda^\Lambda$ is a bijection between Λ^n and W_n^Λ for all $n \in \mathbb{N}^2$.

Proposition 4.3. *Let (Λ, d) be a finite 2-graph with no sources, and let $M_1^{\mathbf{1}\Lambda}$ and $M_2^{\mathbf{1}\Lambda}$ be the matrices associated to $\mathbf{1}\Lambda$. Then*

1. $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ *satisfy* (H0), (H1a), and (H1b);
2. $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H2) if and only if Λ is strongly connected; and
3. if $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H2), then they satisfy (H3) if and only if Λ has an aperiodic infinite path.

Proof. For (1), note that each $M_i^{\mathbf{1}\Lambda}$ is a finite square matrix over $\mathbf{1}\Lambda^0$ by definition, and has entries in $\{0, 1\}$ by Corollary 3.10. It is easy to see that

$$(M_i^{\mathbf{1}\Lambda} M_{3-i}^{\mathbf{1}\Lambda})_{v,w} = |\{(\alpha, \beta) \in w(\mathbf{1}\Lambda^{e_{3-i}}) \times (\mathbf{1}\Lambda^{e_i})v : r(\alpha) = s(\beta)\}| = |w(\mathbf{1}\Lambda^{\mathbf{1}})v|$$

for $i = 1, 2$, which establishes (H1a) and, when combined with Lemma 3.7, (H1b).

For (2), notice that $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H2) if and only if for every $v, w \in \mathbf{1}\Lambda^0$ there exist elements $\alpha_1, \ldots, \alpha_k$ in $\mathbf{1}\Lambda^{(1,0)} \cup \mathbf{1}\Lambda^{(0,1)}$ such that $r(\alpha_1) = v$, $s(\alpha_k) = w$, and $r(\alpha_{i+1}) = s(\alpha_i)$ for $1 \le i \le k - 1$.

So suppose first that $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H2), and let $v, w \in \Lambda^0$. Since Λ has no sources, there exist $\mu, \nu \in \Lambda^{\mathbf{1}}$ with $r(\mu) = v$ and $r(\nu) = w$; so $\mu, \nu \in \mathbf{1}\Lambda^0$ by definition, and (H2) ensures that there is a path $\alpha_1 \dots \alpha_k$ in $\mathbf{1}\Lambda^{(1,0)} \cup \mathbf{1}\Lambda^{(0,1)}$ with $r_{\mathbf{1}}(\alpha_1) = \mu$ and $s_{\mathbf{1}}(\alpha_k) = \nu$. By definition of $\mathbf{1}\Lambda$, the path $\alpha_1 \dots \alpha_k$ in $\mathbf{1}\Lambda$ is a path $\lambda \in \Lambda$ with $d(\lambda) = d_{\mathbf{1}}(\alpha_1 \dots \alpha_k) + \mathbf{1}$, and such that $\lambda(0, \mathbf{1}) = \mu$ and $\lambda(d(\lambda) - \mathbf{1}, d(\lambda)) = \nu$. But then $\lambda(0, d(\lambda) - \mathbf{1}) \in v\Lambda w$. But $v, w \in \Lambda^0$ were arbitrary, so Λ is strongly connected.

Now suppose that Λ is strongly connected, and fix $\mu, \nu \in \mathbf{1}\Lambda^0$. Since Λ is strongly connected, there is a path $\lambda \in s(\mu)\Lambda r(\nu)$, and then $\tau := \mu\lambda\nu$ belongs to $\mu(\mathbf{1}\Lambda)\nu$ with $d_{\mathbf{1}}(\mu\lambda\nu) = d(\lambda) + \mathbf{1}$. Any factorisation of τ into segments from $\mathbf{1}\Lambda^{(1,0)} \cup \mathbf{1}\Lambda^{(0,1)}$ now gives a path in $\mathbf{1}\Lambda^{(1,0)} \cup \mathbf{1}\Lambda^{(0,1)}$ from ν to μ, so $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H2).

Finally, for (3), assume that $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H2), so Λ is strongly connected by part (2). For $x \in \Lambda^\infty$, define $\mathbf{1}x \in \mathbf{1}\Lambda^\infty$ by $(\mathbf{1}x)(m, n) = x(m, n + \mathbf{1})$. It is easy to see that the map $x \mapsto \mathbf{1}x$ is a bijection between Λ^∞ and $\mathbf{1}\Lambda^\infty$.

Claim: $x \in \Lambda^\infty$ is aperiodic if and only if $\mathbf{1}x \in \mathbf{1}\Lambda^\infty$ is aperiodic. To see this, let $m, n \in \mathbb{N}^k$, and fix $x \in \Lambda^\infty$. By definition, we have

$$\begin{aligned}\sigma^m(\mathbf{1}x) = \sigma^n(\mathbf{1}x) &\iff (\mathbf{1}x)(s+m, t+m) = (\mathbf{1}x)(s+n, t+n) \quad \text{for } s \le t \\ &\iff x(s+m, t+m+\mathbf{1}) = x(s+n, t+n+\mathbf{1}) \quad \text{for } s \le t \end{aligned} \tag{4.4}$$

Now if $x(s+m, t+m+\mathbf{1}) = x(s+n, t+n+\mathbf{1})$ for all $s \le t \in \mathbb{N}^2$, then the uniqueness of factorisations in Λ ensures that $x(s+m, t+m) = x(s+n, t+n)$ for all $s \le t \in \mathbb{N}^2$. Conversely if $x(s+m, t+m) = x(s+n, t+n)$ for all $s \le t \in \mathbb{N}^2$, then replacing t with $t + \mathbf{1}$ gives $x(s+m, t+m+\mathbf{1}) = x(s+n, t+n+\mathbf{1})$ for all $s \le t \in \mathbb{N}^2$. Hence (4.4) shows that

$$\begin{aligned}\sigma^m(\mathbf{1}x) = \sigma^n(\mathbf{1}x) &\iff x(s+m, t+m) = x(s+n, t+n) \quad \text{for } s \le t \in \mathbb{N}^2 \\ &\iff \sigma^m(x) = \sigma^n(x),\end{aligned}$$

establishing the claim. Thus it suffices to show that $M_i^{\mathbf{1}\Lambda}$ satisfy (H3) if and only if $\mathbf{1}\Lambda^\infty$ has an aperiodic element.

Suppose first that there exists an aperiodic path $x \in \mathbf{1}\Lambda^\infty$. Fix $m \in \mathbb{Z}^2$, and write $m = m_+ - m_-$ where $m_+, m_- \in \mathbb{N}^2$. Since $|v(\mathbf{1}\Lambda^{e_i})w| \in \{0, 1\}$ for all $v, w \in \mathbf{1}\Lambda^0$, $i = 1, 2$, we have that x is completely determined by its restriction to the objects of Ω_2; that is, by the function from $\mathbb{N}^2$ to $\mathbf{1}\Lambda^0$ given by $n \mapsto x(n)$. Since x is aperiodic, it follows that $\sigma^{m_+}(x)(n) \ne \sigma^{m_-}(x)(n)$ for some $n \in \mathbb{N}^2$. But then with $N := n + m_-$, we have $x(N + m_+ - m_-) \ne x(N)$, and $w := x|_{[0, N+m_+-m_-]} \in W^{\mathbf{1}\Lambda}_{N+m_+-m_-}$ satisfies $w(N) \ne w(N + m)$. Since $m \in \mathbb{Z}^2$ was arbitrary, this establishes that $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H3).

Now suppose that $M_1^{\mathbf{1}\Lambda}, M_2^{\mathbf{1}\Lambda}$ satisfy (H3). For each $m \in \mathbb{Z}^2 \setminus \{0\}$, fix $w_m \in W^{\mathbf{1}\Lambda}$ and $l_m \in \mathbb{N}^2$ such that $0 \le l_m, l_m + m \le S(w_m)$ and $w_m(l_m) \ne w_m(l_m + m)$. Let λ_m be the unique path in $\mathbf{1}\Lambda$ such that $w_m = w^{\mathbf{1}\Lambda}_{\lambda_m}$. We will construct an infinite path x which contains infinitely many occurrences of each λ_m; this will ensure that there is no m for which a sufficiently large shift of x has period m, and hence that x is aperiodic. The details of this construction, and the verification that the resulting x is aperiodic constitute the remainder of the proof.

Let $\{m_i : i \in \mathbb{N}\}$ be a listing of $\mathbb{Z}^2 \setminus \{0\}$. Fix an arbitrary $v \in \mathbf{1}\Lambda^0$, and for each $i \in \mathbb{N}$, let α_i be any element of $v(\mathbf{1}\Lambda)r(\lambda_{m_i})$, and let β_i be any element of $s(\lambda_{m_i})(\mathbf{1}\Lambda)v$ with the property that $d_{\mathbf{1}}(\alpha_i \lambda_{m_i} \beta_i) \ge \mathbf{1}$; this is possible because Λ is strongly connected and has no sources.

For $i \in \mathbb{N}$, let $\rho_i = \alpha_i \lambda_{m_i} \beta_i$, and let $\tau_i = \rho_1 \rho_2 \dots \rho_i$. Let x be the infinite path $x = \tau_1 \tau_2 \tau_3 \cdots$. We claim that x is aperiodic.

To see this, let $s, t \in \mathbb{N}^2$ be distinct, and let $I_{s,t}$ be the element of $\mathbb{N}$ such that $m_{I_{s,t}} = t - s$. Let $J = \max\{s_1, s_2, t_1, t_2\}$; since $d_{\mathbf{1}}(\rho_i) \geq (1, 1)$, we have that $i \geq J$ implies $d_{\mathbf{1}}(\tau_1 \cdots \tau_i) \geq s, t$. Let $K = \max\{I_{s,t}, J + 1\}$, and define $N = d_{\mathbf{1}}(\tau_1 \cdots \tau_{K-1}) + d_{\mathbf{1}}(\rho_1 \cdots \rho_{I_{s,t}-1}) + d(\alpha_{I_{s,t}}) + l_{t-s} - s$. We have $N \geq 0$ by choice of K, and

$$\begin{aligned}\sigma^s(x)(N) &= x(N + s)\\ &= x(d_{\mathbf{1}}(\tau_1 \cdots \tau_{K-1}) + d_{\mathbf{1}}(\rho_1 \cdots \rho_{I_{s,t}-1}) + d(\alpha_{I_{s,t}}) + l_{t-s})\\ &= \lambda_{m_{I_{s,t}}}(l_{t-s}).\end{aligned}$$

A similar calculation shows that $\sigma^t(x)(N) = \lambda_{m_{I_{s,t}}}(l_{t-s} + (t - s))$, and hence $\sigma^s(x)(N) \neq \sigma^t(x)(N)$ by our chice of $\lambda_{m_{I_{s,t}}}$. It follows that $\sigma^s(x) \neq \sigma^t(x)$, and since $s, t \in \mathbb{N}^2$ were arbitrary, that x is aperiodic. □

Remark 4.4. The preceding proof actually shows: (1) that Proposition 4.3 applies to M_i^Λ for any Λ satisfying the property of Lemma 3.7 for $p = \mathbf{1}$; and (2) that $\mathbf{1}\Lambda$ is strongly connected (resp. strongly connected and contains an infinite path) if and only if Λ has the same property. Since our motivation is to prove Theorem 4.1, we have compressed this into a single result.

Notation 4.5. Let Λ be a finite strongly connected 2-graph with an aperiodic infinite path. We write $\mathcal{A}^{\mathbf{1}\Lambda}$ for the C^*-algebra associated to $M_i^{\mathbf{1}\Lambda}$ as in [16]. That is, $\mathcal{A}^{\mathbf{1}\Lambda}$ is the universal C^*-algebra generated by a family $\{s_{u,v} : u, v \in W^{\mathbf{1}\Lambda}, u(S(u)) = v(S(v))\}$ of partial isometries satisfying

$$s_{u,v} = s_{v,u}^* \quad \text{for } u, v \in W^{\mathbf{1}\Lambda}; \tag{4.5}$$

$$s_{u,v} s_{v,w} = s_{u,w} \quad \text{for } u, v, w \in W^{\mathbf{1}\Lambda}; \tag{4.6}$$

$$s_{u,v} = \sum_{w \in W^{\mathbf{1}\Lambda}_{e_j}, u(S(u)) = w(0)} s_{uw} s_{vw}^* \quad \text{for } u, v, \in W^{\mathbf{1}\Lambda},\ j \in \{1, 2\}; \text{ and} \tag{4.7}$$

$$s_{a,a} s_{b,b} = 0 \quad \text{for distinct } a, b \in W_0^{\mathbf{1}\Lambda}. \tag{4.8}$$

Lemma 4.6. *Let (Λ, d) be a finite strongly connected 2-graph which has an aperiodic infinite path. Then $C^*(\Lambda)$ is isomorphic to $\mathcal{A}^{\mathbf{1}\Lambda}$.*

Proof. The factorisation property ensures that if Λ is strongly connected and contains an infinite path, then Λ has no sources. By Theorem 3.5, we have that $C^*(\Lambda)$ is isomorphic to $C^*(\mathbf{1}\Lambda)$, so it suffices to show that $C^*(\mathbf{1}\Lambda)$ is isomorphic to $\mathcal{A}^{\mathbf{1}\Lambda}$. It is easy to check using Definition 2.4(i)–(iv), relations (4.5)–(4.8), and the universal properties of $\mathcal{A}^{\mathbf{1}\Lambda}$ and $C^*(\mathbf{1}\Lambda)$ that there exists a homomorphism $\pi : \mathcal{A}^{\mathbf{1}\Lambda} \to C^*(\mathbf{1}\Lambda)$ satisfying $\pi(s_{w_\lambda^{\mathbf{1}\Lambda}, w_\mu^{\mathbf{1}\Lambda}}) = s_\lambda s_\mu^*$ for all $\lambda, \mu \in \mathbf{1}\Lambda$, and that there exists a homomorphism $\psi : C^*(\mathbf{1}\Lambda) \to \mathcal{A}^{\mathbf{1}\Lambda}$ satisfying $\psi(s_\lambda) = s_{w_\lambda^{\mathbf{1}\Lambda}, w_{s(\lambda)}^{\mathbf{1}\Lambda}}$. Since these two homomorphisms are mutually inverse, the result follows. □

Remark 4.7. The argument of statement (2) of Proposition 4.3 shows that if Λ has no sources, then for any $q \geq \mathbf{1}$, the coordinate matrices of $q\Lambda$ will satisfy (H2) only if Λ is strongly connected and has no sources. In particular, there exists $q \in \mathbb{N}^2$ such that $M_i^{q\Lambda}$ satisfy (H0)–(H3) if and only if $M_i^{\mathbf{1}\Lambda}$ satisfy (H0)–(H3).

Proof of Theorem 4.1. Theorem 5.9, Proposition 5.11, and Corollary 6.4 of [15] combined with the previous two results show that $C^*(\Lambda)$ is simple, purely infinite and nuclear. We have that $C^*(\Lambda)$ is unital with $1_{C^*(\Lambda)} = \sum_{v\in\Lambda^0} s_v$. Proposition 2.14 of [16] establishes (4.1)–(4.3). □

Remarks 4.8. (1) The proof of [16, Proposition 2.14] does not make any use of relations (H2) and (H3). Hence the formulas for $K_*(C^*(\Lambda))$ in Theorem 4.1 hold when Λ is a finite k-graph with no sinks or sources, even if it is not strongly connected and does not have an aperiodic infinite path. However, in this case $C^*(\Lambda)$ is not necessarily simple and purely infinite, and so is not determined up to isomorphism by its K-theory.

(2) The formulas for $K_*(C^*(\Lambda))$ given in Theorem 4.1 are in terms of the coordinate matrices $M_i^{\mathbf{1}\Lambda}$ of the dual k-graph. Proposition 5.1 of [6] shows that the same formulas hold if all instances $M_i^{\mathbf{1}\Lambda}$ are replaced with M_i^{Λ}, but it is unclear how to show this directly.

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