A DUAL GRAPH CONSTRUCTION FOR HIGHER-RANK GRAPHS, AND \( K \)-THEORY FOR FINITE 2-GRAPHS

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Abstract. Given a \( k \)-graph \( \Lambda \) and an element \( p \) of \( \mathbb{N}^k \), we define the dual \( k \)-graph, \( p\Lambda \). We show that when \( \Lambda \) is row-finite and has no sources, the \( C^* \)-algebras \( C^*(\Lambda) \) and \( C^*(p\Lambda) \) coincide. We use this isomorphism to apply Robertson and Steger’s results to calculate the \( K \)-theory of \( C^*(\Lambda) \) when \( \Lambda \) is finite and strongly connected and satisfies the aperiodicity condition.

1. Introduction

In 1980, Cuntz and Krieger introduced a class of \( C^* \)-algebras, now called Cuntz-Krieger algebras, associated to finite \( \{0,1\} \)-matrices \( A \) [4]. Enomoto and Watatani then showed that these algebras could be regarded as being associated in a natural way to finite directed graphs by regarding \( A \) as the vertex adjacency matrix of a finite directed graph \( E \) [5]. Generalising this association, Enomoto and Watatini associated \( C^* \)-algebras \( C^*(E) \) to finite graphs \( E \) with no sources\(^1 \) (\( E \) has no sources if each vertex of \( E \) is the range of at least one edge). Although not every finite directed graph with no sources has a vertex adjacency matrix with entries in \( \{0,1\} \), the vertex adjacency matrix of the dual graph \( \hat{E} \) formed by regarding the edges of \( E \) as vertices and the paths of length 2 in \( E \) as edges does always have entries in \( \{0,1\} \), and the Cuntz-Krieger algebras associated to \( E \) and to \( \hat{E} \) are canonically isomorphic [11]. These results have since been extended to infinite graphs (see for example [10, 9, 3, 7]; see also [2] when \( E \) has sources).

One of the major attractions of graph algebras is their applicability to the classification program for simple purely infinite nuclear \( C^* \)-algebras. Conditions on a graph \( E \) have been identified which guarantee that \( C^*(E) \) is purely infinite, simple, and nuclear, and satisfies the Universal Coefficient Theorem (see, for example, [3]), thus producing a large class of directed graphs whose \( C^* \)-algebras are determined up to isomorphism by their \( K \)-theory [12]. The \( K \)-theory of \( C^*(E) \) for an arbitrary directed graph \( E \) was calculated in [13], and it is shown in [17] that given any two finitely generated abelian groups \( G, H \) such that \( H \) is torsion-free, there exists a directed graph \( E \) such that \( C^*(E) \) is simple, purely infinite, nuclear, and satisfies the Universal Coefficient Theorem, with \( K_0(C^*(E)) \cong G \) and \( K_1(C^*(E)) \cong H \).

\(^{1}\)For the sake of consistency with \( k \)-graph notation, we regard directed graphs as 1-graphs, so no sources here corresponds to no sinks in, for example, [5, 3]

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In 1999, Robertson and Steger introduced a class of higher-rank Cuntz-Krieger algebras \( \mathcal{A} \), associated to collections \( M_1, \ldots, M_k \) of commuting \( \{0,1\}\)-matrices satisfying appropriate compatibility conditions \([15]\). In \([16]\), they went on to calculate the \( K \)-theory of \( \mathcal{A} \), demonstrating in particular that \( K_1(\mathcal{A}) \) need not be torsion-free, so that the class of higher-rank Cuntz-Krieger algebras exhausts some \( K \)-invariants which are not achieved by graph algebras. In order to place these higher-rank Cuntz-Krieger algebras in a graph-theoretic setting, and to generalise them as Watatani and Enomoto had generalised the original Cuntz-Krieger algebras, Kumjian and Pask introduced the notion of a higher-rank graph \( \Lambda \), and defined and investigated the associated higher-rank graph \( C^* \)-algebra \( C^*(\Lambda) \) \([8]\).

Connectivity in a rank-\( k \) graph \( \Lambda \) is described in terms of \( k \) commuting vertex adjacency matrices \( \{M^{\Lambda}_1, \ldots, M^{\Lambda}_k\} \), called coordinate matrices. Just as in the rank-1 setting, not every \( k \)-graph has coordinate matrices with entries in \( \{0,1\} \), but if \( \Lambda \) is a \( k \)-graph whose coordinate matrices are \( \{0,1\}\)-matrices, then \([8, \text{Corollary 3.5}(ii)]\) shows that \( C^*(\Lambda) \) and the \( C^* \)-algebra \( \mathcal{A} \) associated to the coordinate matrices as in \([15]\) are identical.

In this paper we introduce a notion of a dual graph for higher-rank graphs, and show that for a large class of higher-rank graphs \( \Lambda \), the dual higher-rank graph \( p\Lambda \) and the original higher-rank graph \( \Lambda \) have canonically isomorphic \( C^* \)-algebras for all \( p \in \mathbb{N}^k \) (c.f. \([1]\)). We also show that by choosing \( p \) appropriately, we can ensure that \( p\Lambda \) has coordinate matrices with entries in \( \{0,1\} \). Using these results, we identify a class of finite rank-2 graphs whose \( C^* \)-algebras are isomorphic to the rank-2 Cuntz-Krieger algebras studied by Robertson and Steger, and we use the results of \([16]\) to show that these \( C^* \)-algebras are purely infinite, simple, unital and nuclear, and to calculate their \( K \)-theory.

The layout of the paper is as follows: in Section 2, we recall the definition of \( k \)-graphs and the associated notation; in Section 3, we introduce the dual graph construction for \( k \)-graphs, and show that this construction preserves the associated \( C^* \)-algebra; and in Section 4, we identify the finite 2-graphs \( \Lambda \) whose \( C^* \)-algebras can be studied using Robertson and Steger’s results, and use these results to calculate \( K_* (C^*(\Lambda)) \).

In the final stages of preparation of this paper, the authors became aware of Evans’ Ph.D. thesis \([6]\), which appears to obtain more general results regarding \( K \)-theory for 2-graph \( C^* \)-algebras than those established here. The authors thank the referee for helpful comments which we feel have improved the exposition.

## 2. Preliminaries

We regard \( \mathbb{N}^k \) as an additive semigroup with identity 0. Given \( m, n \in \mathbb{N}^k \), we write \( m \vee n \) for their coordinate-wise maximum and \( m \wedge n \) for their coordinate-wise minimum, and if \( m \leq n \), then we write \([m, n]\) for the set \( \{p \in \mathbb{N}^k : m \leq p \leq n\} \). We denote the canonical generators of \( \mathbb{N}^k \) by \( \{e_1, \ldots, e_k\} \), and for \( n \in \mathbb{N}^k \), we write \( n_j \) for the \( j^{\text{th}} \) coordinate of \( n \).

**Definition 2.1.** Let \( k \in \mathbb{N} \setminus \{0\} \). A \( k \)-graph is a pair \( (\Lambda, d) \) where \( \Lambda \) is a countable category and \( d \) is a functor from \( \Lambda \) to \( \mathbb{N}^k \) which satisfies the factorisation property: if \( \lambda \in \text{Mor}(\Lambda) \) and \( d(\lambda) = m + n \), then there are unique morphisms \( \mu \in d^{-1}(m) \) and \( \nu \in d^{-1}(n) \) such that \( \lambda = \mu \nu \).
We refer to elements of $\text{Mor}(\Lambda)$ as paths and to elements of $\text{Obj}(\Lambda)$ as vertices and we write $r$ and $s$ for the codomain and domain maps. The factorisation property allows us to identify $\text{Obj}(\Lambda)$ with $\{\lambda \in \text{Mor}(\Lambda) : d(\lambda) = 0\}$. So we write $\lambda \in \Lambda$ in place of $\lambda \in \text{Mor}(\Lambda)$, and when $d(\lambda) = 0$, we regard $\lambda$ as a vertex of $\Lambda$.

Given $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we define $\lambda E = \{\lambda \mu : \mu \in E, r(\mu) = s(\lambda)\}$ and $E\lambda = \{\mu \lambda : \mu \in E, s(\mu) = r(\lambda)\}$. In particular if $d(\nu) = 0$, then $\nu$ is a vertex of $\Lambda$ and $vE = \{\lambda \in E : r(\lambda) = v\}$; similarly, $Ev = \{\lambda \in \Lambda : s(\lambda) = v\}$. We write $\Lambda^n$ for the collection $\{\lambda \in \Lambda : d(\lambda) = n\}$.

**Definition 2.2.** We say that a $k$-graph $(\Lambda, d)$ is row-finite if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and that $\Lambda$ has no sources if $\nu\Lambda^n$ is nonempty for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that $\Lambda$ is strongly connected if $\nu\Lambda^\omega$ is nonempty for all $v, w \in \Lambda^0$, and we say that $\Lambda$ is finite if $\Lambda^0$ and each $\Lambda^\omega$ are finite.

The factorisation property ensures that if $l \leq m \leq n \in \mathbb{N}^k$ and if $d(\lambda) = n$, then there exist unique paths denoted $\lambda(0, l)$, $\lambda(l, m)$ and $\lambda(m, n)$ such that $d(\lambda(0, l)) = l$, $d(\lambda(l, m)) = m - l$, and $d(\lambda(m, n)) = n - m$ and such that $\lambda = \lambda(0, l)\lambda(l, m)\lambda(m, n)$.

Given $k \in \mathbb{N} \setminus \{0\}$, and $k$-graphs $(\Lambda_1, d_1)$ and $(\Lambda_2, d_2)$, we call a covariant functor $x : \Lambda_1 \to \Lambda_2$ a graph morphism if it satisfies $d_2 \circ x = d_1$.

**Definition 2.3.** As in [8], given $k \in \mathbb{N} \setminus \{0\}$, we write $\Omega_k$ for the $k$-graph given by

$$\text{Obj}(\Omega_k) = \mathbb{N}^k, \text{Mor}(\Omega_k) = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\},$$

so that $\lambda(0, l)$, $\lambda(l, m)$ and $\lambda(m, n)$ such that $d(\lambda(0, l)) = l$, $d(\lambda(l, m)) = m - l$, and $d(\lambda(m, n)) = n - m$ and such that $\lambda = \lambda(0, l)\lambda(l, m)\lambda(m, n)$.

Given $k \in \mathbb{N} \setminus \{0\}$, and $k$-graphs $(\Lambda_1, d_1)$ and $(\Lambda_2, d_2)$, we call a covariant functor $x : \Lambda_1 \to \Lambda_2$ a graph morphism if it satisfies $d_2 \circ x = d_1$.

**Definition 2.4.** Let $(\Lambda, d)$ be a row-finite $k$-graph with no sources. A Cuntz-Krieger $\Lambda$-family is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

1. $\{t_\lambda : \lambda \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
2. $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
3. $t_{\lambda}^* t_{\lambda} = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
4. $t_\lambda = \sum_{\nu \in \text{Vol} \lambda} t_{\nu}\lambda$ for all $\nu \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The Cuntz-Krieger algebra $C^*(\Lambda)$ is the $C^*$-algebra generated by a Cuntz-Krieger $\Lambda$-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the sense that for every Cuntz-Krieger $\Lambda$-family $\{t_\lambda : \lambda \in \Lambda\}$ there is a unique homomorphism $\pi$ of $C^*(\Lambda)$ satisfying $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

### 3. Dual Higher Rank Graphs

In this section we define the higher rank analog $p\Lambda$ of the dual graph construction for directed graphs.

**Definition 3.1.** Let $(\Lambda, d)$ be a $k$-graph and let $p \in \mathbb{N}^k$. Let $p\Lambda = \{\lambda \in \Lambda : d(\lambda) \geq p\}$. Define range and source maps on $p\Lambda$ by $r_p(\lambda) = \lambda(0, p)$, and $s_p(\lambda) = \lambda(e(d(\lambda) - p, d(\lambda)))$ for all $\lambda \in p\Lambda$, and define composition by $\lambda_2 \circ_p \lambda_1 = \lambda_2(0, d(\lambda_1) - p)\mu$ whenever $s_p(\lambda_1) = r_p(\lambda_2)$. Finally, define a degree map $d_p$ on $p\Lambda$ by $d_p(\lambda) = d(\lambda) - p$ for all $\lambda \in p\Lambda$.

**Proposition 3.2.** Let $(\Lambda, d)$ be a $k$-graph, and let $p \in \mathbb{N}^k$. Then $(p\Lambda, d_p)$ is a $k$-graph.
Proof. It is straightforward to check that $p\Lambda$ is a category with the indicated operations. If $\lambda, \mu \in p\Lambda$ and $s_\gamma(\lambda) = r_\nu(\mu)$, then $\lambda \circ_p \mu = \lambda\mu(p, d(\mu))$ by definition, so $d_\rho(\lambda \circ_p \mu) = d(\lambda) + d(\mu) - 2p = d_\rho(\lambda) + d_\rho(\mu)$. So $d_\rho$ is a functor from $p\Lambda$ to $\mathbb{N}^k$.

We need to check that the factorisation property holds for $p\Lambda$. Take any $\lambda \in p\Lambda$ and $m, n \in \mathbb{N}^k$ with $m + n = d_\rho(\lambda)$, so $d(\lambda) = m + p + n$. By the factorisation property for $\Lambda$ we have $\lambda = \lambda(0, m)\lambda(m, m + p)\lambda(m + p, m + n)$. But then $\lambda = (\lambda(0, m)\lambda(m, m + p)) \circ_p (\lambda(m, m + p)\lambda(m + p, m + n))$ in $p\Lambda$, and $d_\rho(\lambda(0, m)\lambda(m, m + p)) = m$ and $d_\rho(\lambda(m, m + p)\lambda(m + p, m + n)) = n$. This decomposition is unique by the factorisation property for $\Lambda$. □

Remark 3.3. If $\Lambda$ has no sources, then $p\Lambda$ has no sources, and if $\Lambda$ is row-finite, then $p\Lambda$ is row-finite.

Proposition 3.4. Let $(\Lambda, d)$ be a $k$-graph, and let $p, q \in \mathbb{N}^k$. Then $q(p\Lambda) = (q+p)\Lambda$.

Proof. By definition, we have $q(p\Lambda)^n = (p\Lambda)^{(n+q)} = \Lambda^{(n+q+p)} = (q+p)\Lambda^n$ for all $n \in \mathbb{N}$. Hence $q(p\Lambda)$ and $(q+p)\Lambda$ have identical elements. For the remainder of the proof, we write $s_q^{p\Lambda}, r_q^{p\Lambda}, \lambda_q^{p\Lambda}$, and $d_q^{p\Lambda}$ for the source, range, composition and degree maps of the dual graph $q(p\Lambda)$.

Fix $\lambda \in \Lambda^{n+p+q}$. We have $s_{q+p}(\lambda) = \lambda(n, n+p+q)$ by definition, while $s_q^{p\Lambda}(\lambda)$ is the final segment $\mu$ of $\lambda$ such that $d(\mu) - p = d_\rho(\mu) = q$; that is $d(\mu) = p + q$. Hence $s_{p+q}(\lambda) = s_q^{p\Lambda}(\lambda)$. Similarly, $r_{p+q}(\lambda) = \lambda(0, p + q) = r_q^{p\Lambda}(\lambda)$. Moreover, $d_{p+q}(\lambda) = d(\lambda) - (p + q) = d_\rho(\lambda) - q = d_q^{p\Lambda}(\lambda)$. Since $\lambda$ was arbitrary, it follows that the range, source, and degree maps for $(p+q)\Lambda$ and $q(p\Lambda)$ agree.

This established, we have $r_{p+q}(\lambda) = s_{p+q}(\mu)$ if and only if $r_q^{p\Lambda}(\lambda) = s_q^{p\Lambda}(\mu)$, in which case both $\lambda \circ_p \mu$ and $\lambda \circ_q p\Lambda$ are equal to $\lambda\mu(p + q, d(\mu))$ by definition, completing the proof. □

Theorem 3.5. Let $(\Lambda, d)$ be a row-finite $k$-graph with no sources, and let $p \in \mathbb{N}^k$. Let $\{s_\lambda : \lambda \in \Lambda\}$ denote the universal generating Cuntz-Krieger $\Lambda$-family in $C^*(\Lambda)$, and let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal generating Cuntz-Krieger $p\Lambda$-family in $C^*(p\Lambda)$. For all $\lambda \in p\Lambda$, define $r_\lambda = s_\lambda s_{p\Lambda}^*(\lambda)$: There is an isomorphism $\phi : C^*(p\Lambda) \to C^*(\Lambda)$ such that $\phi(t_\lambda) = r_\lambda$ for all $\lambda \in p\Lambda$.

Proof. First we show that the family $\{r_\lambda : \lambda \in p\Lambda\}$ is a Cuntz-Krieger $p\Lambda$-family. Since, for any $\beta \in p\Lambda^0$, we have $s_{\beta} \neq 0$ it follows that $r_{\beta} = s_{\beta}s_{p\Lambda}^* \neq 0$ and that it is a projection in $C^*(\Lambda)$. Furthermore, for distinct $\alpha, \beta \in p\Lambda^0$, we have

$$r_\alpha r_\beta = s_\alpha s_{p\Lambda}^* s_\beta = \delta_{\alpha, p\Lambda} s_{p\Lambda} = \delta_{\alpha, p\Lambda} r_\alpha.$$  

This establishes relation (i).

For relation (ii), let $\mu, \nu \in p\Lambda$ with $r_\mu(\nu) = s_\nu(\mu)$, so $\mu \circ_p \nu = \mu\nu(p, d(\nu))$. Then,

$$r_{\mu \circ_p \nu} = s_{\mu \circ_p \nu} s_{p\Lambda}^*(\mu \circ_p \nu) = s_\mu s_{\nu(p, d(\nu))} s_{p\Lambda}^*(\nu) = s_\mu s_{p\Lambda}^*(\nu) s_{p\Lambda}^*(\nu) = r_\mu r_\nu.$$  

But $s_\mu(\nu) = r_\mu(\nu) = \nu(0, p)$, so we can rewrite the right-hand side of (3.1) to obtain

$$r_{\mu \circ_p \nu} = s_{\mu} s_{p\Lambda}^*(\nu) = r_\mu r_\nu.$$  

This establishes relation (ii).

Let $\lambda \in p\Lambda$, say $d_\rho(\lambda) = n$. Then $r_\lambda s_{p\Lambda} = s_{p\Lambda}(\lambda) s_{p\Lambda} s_{p\Lambda}(\lambda) = s_{p\Lambda}(\lambda) s_{p\Lambda}(\lambda) = r_\lambda s_{p\Lambda}$ by definition, establishing relation (iii).

Finally, for relation (iv), let $\beta \in p\Lambda^0$ and let $n \in \mathbb{N}^k$. Then

$$r_\beta = s_\beta s_{p\Lambda}^* = \sum_{\gamma \in s(\beta)p\Lambda^0} s_{\gamma} s_{p\Lambda}^* = \sum_{\lambda \in p\Lambda^0} s_{\lambda} s_{p\Lambda}^*.$$
Applying the factorisation property and relation (ii) for \( C^*(\Lambda) \) to the right-hand side then gives

\[
r_\beta = \sum_{\lambda \in \beta \Lambda^s} s_{\lambda(0,n)} s_{\lambda(n,n+p)}^* s_{\lambda(n,n+p)}^* s_{\lambda(n,n+p)} s_{\lambda(0,n)},
\]

and then since each \( s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^* \) is a projection, we obtain

\[
r_\beta = \sum_{\lambda \in \beta \Lambda^s} (s_{\lambda(0,n)} s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^*) (s_{\lambda(n,n+p)} s_{\lambda(n,n+p)}^* s_{\lambda(n,n+p)} s_{\lambda(0,n)}) = \sum_{\lambda \in \beta(p\Lambda^n)} r_\lambda r_\lambda^*,
\]

which establishes relation (iv).

It follows from the universal property of \( C^*(p\Lambda) \) that there exists a homomorphism \( \phi : C^*(p\Lambda) \to C^*(\Lambda) \) satisfying \( \phi(t_\lambda) = r_\lambda \) for all \( \lambda \in p\Lambda \). We claim that \( \{t_\lambda : \lambda \in p\Lambda\} \) generates \( C^*(\Lambda) \). To see this, let \( \sigma \in \Lambda \) with \( d(\sigma) = n \). An application of relation (iv) for \( C^*(\Lambda) \) gives \( s_\sigma = \sum_{p \in \sigma(\Lambda)^p} s_{\sigma} s_{\sigma}^* = \sum_{p \in \sigma(\Lambda)^p} s_{\lambda} s_{\lambda}^* \), and this last is equal to \( \sum_{\lambda \in \sigma(\Lambda)^n} r_\lambda \), by definition. Thus \( \phi \) maps \( C^*(p\Lambda) \) onto \( C^*(\Lambda) \).

Now let \( \gamma^\Lambda \) denote the gauge action on \( C^*(\Lambda) \), and let \( \gamma^\Lambda \) denote the gauge action on \( C^*(p\Lambda) \). For \( z \in \mathbb{T}^\Lambda \) and \( \lambda \in p\Lambda \), we have

\[
\gamma^\Lambda_z(r_\lambda) = \gamma^\Lambda_z(s_{\lambda} s_{\lambda}^*(\Lambda)) = z^d(\lambda) s_{\lambda} (z^d(s_{\lambda}^*(\Lambda)))^* = z^d(\lambda)^{-1} r_\lambda = \gamma^\Lambda(r_\lambda).
\]

Theorem 3.4 of [8] now establishes that \( \phi \) is injective. \( \square \)

**Remark 3.6.** The hypotheses that \( \Lambda \) be row-finite and have no sources are crucial in Theorem 3.5. To see why, notice that for \( v \in \Lambda^0 \), the generator \( s_v \) of \( C^*(\Lambda) \) is recovered in \( C^*(p\Lambda) \) as \( \sum_{\beta \in \Lambda^p, r(\beta) = v} r_\beta \). However, even for 1-graphs the Cuntz-Krieger relations only insist that \( p_v = \sum_{r(\beta) = v} s_{\beta} s_{\beta}^* \) when \( r^{-1}(v) \) is finite and nonempty.

**Lemma 3.7.** Let \((\Lambda, d)\) be a k-graph, and let \( p \in \mathbb{N}^k \). For each \( n \in \mathbb{N} \) with \( n \leq p \) and \( v, w \in p\Lambda^0 \), there is at most one \( \lambda \in v(p\Lambda^n)w \).

**Proof.** Let \( v, w \in p\Lambda^0 = \Lambda^p \) and suppose \( \lambda \in v(p\Lambda^n)w \). Then \( \lambda \in \Lambda^{n+p}, \lambda(0, p) = v \), and \( \lambda(n, n + p) = w \). Since \( n \leq p \) we have \( \lambda(0, n) = \{\lambda(0, p)\}(0, n) = v(0, n) \), so \( \lambda = \lambda(0, n) \lambda(n, n + p) = v(0, n)w \), and hence is determined by \( v \) and \( w \). \( \square \)

**Notation 3.8.** Let \((\Lambda, d)\) be a k-graph. We write \( M^\Lambda_i \), \( 1 \leq i \leq k \) for the matrices in \( M_{\Lambda^0}(\mathbb{N}) \) defined by \( (M^\Lambda_i)_{v, w} = |v\Lambda^e w| \) for \( v, w \in \Lambda^0 \), and we refer to these matrices as the coordinate matrices of \( \Lambda \).

**Remark 3.9.** In [8, 6] \( (M^\Lambda)_{v, w} = |v\Lambda^e w| \), so our \( M^\Lambda \) is the transpose of theirs. This is for consistency with the matrices in [16, 15]; we will be employing Robertson and Steger’s results to calculate \( K \)-theory in Section 4.

**Corollary 3.10.** Let \((\Lambda, d)\) be a k-graph, and let and \( p \in \mathbb{N}^k \) with \( p_i \geq 1 \) for \( 1 \leq i \leq k \). Then the coordinate matrices \( M^\Lambda_i \) of \( p\Lambda \) are \( \{0, 1\} \)-matrices.

4. \( K \)-theory

In this section we identify a class of 2-graphs whose associated \( C^* \)-algebras are isomorphic to higher rank Cuntz-Krieger algebras in the sense of [16], and use the results of [16] to calculate the \( K \)-theory of the \( C^* \)-algebras of such 2-graphs. To state the main theorem for this section we employ the following notation: given square \( n \times n \) matrices \( M, N \), we write \( [M \quad N] \) for the block \( n \times 2n \) matrix whose first \( n \) columns are those of \( M \) and whose last \( n \) columns are those of \( N \). We also write \( 1 \) for the element \((1, 1)\) of \( \mathbb{N}^2 \).
Theorem 4.1. Let $(\Lambda, d)$ be a 2-graph which is finite and strongly connected as in Definition 2.2 and which has an aperiodic infinite path as in Definition 2.3. Then $C^*(\Lambda)$ is purely infinite, simple, unital and nuclear, and we have

$$\text{rank}(K_0(C^*(\Lambda))) = \text{rank}(\text{coker} \left[ I - M_1^{1\Lambda} \right. I - M_2^{1\Lambda} \left. \right])$$

$$+ \text{rank}(\text{coker} \left[ I - (M_1^{1\Lambda})^t \quad I - (M_2^{1\Lambda})^t \right]);$$

$$\text{tor}(K_0(C^*(\Lambda))) \cong \text{tor}(\text{coker} \left[ I - M_1^{1\Lambda} \quad I - M_2^{1\Lambda} \right]); \text{ and}$$

$$\text{tor}(K_1(C^*(\Lambda))) \cong \text{tor}(\text{coker} \left[ I - (M_1^{1\Lambda})^t \quad I - (M_2^{1\Lambda})^t \right]).$$

The remainder of this section constitutes the proof of Theorem 4.1. We begin by recalling some definitions from [16]. Let $A$ be a finite set, and let $M_1, M_2$ be $A \times A$ matrices with entries in $\{0, 1\}$. For $n \in \mathbb{N}^+$, let $W_n = \{w : [0, n] \rightarrow A : M_j(w(l + e_j), w(l)) = 1 \text{ whenever } l, l + e_j \in [0, n]\}$; we refer to the elements of $W_n$ as allowable words of shape $n$, and write $W$ for the collection $\bigcup_{n \in \mathbb{N}^2} W_n$ of all allowable words. For $u \in W$, write $S(u)$ for the shape of $u$; that is, $S(u)$ is the unique element of $\mathbb{N}^2$ such that $u \in W_{S(u)}$. We identify $W_0$ with $A$. The matrices $M_1, M_2$ are said to satisfy (H0)–(H3) if

(H0) Each $M_i$ is nonzero;

(H1a) $M_1 M_2 = M_2 M_1$;

(H1b) $M_1 M_2$ is a $\{0, 1\}$-matrix;

(H2) the directed graph with a vertex for each $a \in A$ and a directed edge $(a, i, b)$ from $a$ to $b$ for each $a, i, b$ such that $M_i(b, a) = 1$ is irreducible; and

(H3) for each $m \in \mathbb{Z}^2 \setminus \{0\}$, there exists a word $w \in W$ and elements $l_1, l_2$ of $\mathbb{N}$ such that $l_2 - l_1 = m$ and $w(l_1) \neq w(l_2)$.

Notation 4.2. If $(\Lambda, d)$ is a 2-graph such that the coordinate matrices $M_1^\Lambda$ and $M_2^\Lambda$ are $\{0, 1\}$-matrices, we write $W_\Lambda$ and $W^\Lambda$ for the collection of allowable words of shape $n$ and for the collection of all allowable words respectively. For $\lambda \in \Lambda$, let $w_\lambda$ be the word in $W_{\delta(\lambda)}$ given by $w_\lambda(m) = s(\lambda(0, m))$ for $0 \leq m \leq d(\lambda)$. Since each $M_i^\Lambda$ is a $\{0, 1\}$-matrix, the map $\lambda \mapsto w_\lambda$ is a bijection between $\Lambda^n$ and $W_\Lambda$ for all $n \in \mathbb{N}$.

Proposition 4.3. Let $(\Lambda, d)$ be a finite 2-graph with no sources, and let $M_1^{1\Lambda}$ and $M_2^{1\Lambda}$ be the matrices associated to $1\Lambda$. Then

1. $M_1^{1\Lambda}, M_2^{1\Lambda}$ satisfy (H0), (H1a), and (H1b);

2. $M_1^{1\Lambda}, M_2^{1\Lambda}$ satisfy (H2) if and only if $\Lambda$ is strongly connected; and

3. if $M_1^{1\Lambda}, M_2^{1\Lambda}$ satisfy (H2), then they satisfy (H3) if and only if $\Lambda$ has an aperiodic infinite path.

Proof. For (1), note that each $M_i^{1\Lambda}$ is a finite square matrix over $1\Lambda^0$ by definition, and has entries in $\{0, 1\}$ by Corollary 3.10. It is easy to see that

$$(M_i^{1\Lambda} M_3^{1\Lambda})_{\alpha, \beta} = |\{(\alpha, \beta) \in w(1\Lambda^{e_3 \cdots e_1}) \times (1\Lambda^{e_1}) : r(\alpha) = s(\beta)\}| = |w(1\Lambda^1)_{\alpha}|$$

for $i = 1, 2$, which establishes (H1a) and, when combined with Lemma 3.7, (H1b).

For (2), notice that $M_1^{1\Lambda}, M_2^{1\Lambda}$ satisfy (H2) if and only if for every $w \in 1\Lambda^0$ there exist elements $\alpha_1, \ldots, \alpha_k$ in $1\Lambda^{(1, 0)} \cup 1\Lambda^{(0, 1)}$ such that $r(\alpha_1) = \alpha, s(\alpha_k) = w$, and $r(\alpha_{i+1}) = s(\alpha_i)$ for $1 \leq i \leq k - 1$. 
So suppose first that $M_{1}^{1\Lambda}, M_{2}^{1\Lambda}$ satisfy (H2), and let $v, w \in \Lambda^0$. Since $\Lambda$ has no sources, there exist $\mu, \nu \in \Lambda^1$ with $r(\mu) = v$ and $r(\nu) = w$; so $\mu, \nu \in \Lambda^0$ by definition, and (H2) ensures that there is a path $\alpha_1 \ldots \alpha_k$ in $\Lambda^{1(0)} \cup \Lambda^{0(1)}$ with $r_1(\alpha_1) = \mu$ and $s_1(\alpha_k) = \nu$. By definition of $\Lambda$, the path $\alpha_1 \ldots \alpha_k$ in $\Lambda$ is a path $\lambda \in \Lambda$ with $d(\lambda) = d_1(\alpha_1 \ldots \alpha_k) + 1$, and such that $\lambda(0, 1) = \mu$ and $\lambda(d(\lambda) - 1, d(\lambda)) = \nu$. But then $\lambda(0, d(\lambda) - 1) \in v\Lambda w$. But $v, w \in \Lambda^0$ were arbitrary, so $\Lambda$ is strongly connected.

Now suppose that $\Lambda$ is strongly connected, and fix $\mu, \nu \in \Lambda^0$. Since $\Lambda$ is strongly connected, there is a path $\lambda \in s(\mu)\Lambda r(\nu)$, and then $\tau := \mu\lambda\nu$ belongs to $\mu(\Lambda)\nu$ with $d_2(\mu\lambda\nu) = d(\lambda) + 1$. Any factorisation of $\tau$ into segments from $\Lambda^{1(0)} \cup \Lambda^{0(1)}$ now gives a path in $\Lambda^{1(0)} \cup \Lambda^{0(1)}$ from $\nu$ to $\mu$, so $M_{1}^{1\Lambda}, M_{2}^{1\Lambda}$ satisfy (H2).

Finally, for (3), assume that $M_{1}^{1\Lambda}, M_{2}^{1\Lambda}$ satisfy (H2), so $\Lambda$ is strongly connected by part (2). For $x \in \Lambda^\infty$, define $1_x \in \Lambda^\infty$ by $(1_x)(m, n) = x(m, n + 1)$. It is easy to see that the map $x \mapsto 1_x$ is a bijection between $\Lambda^\infty$ and $\Lambda^{\infty}$.

Claim: $x \in \Lambda^\infty$ is aperiodic if and only if $1_x \in \Lambda^{\infty}$ is aperiodic. To see this, let $m, n \in \mathbb{N}^k$, and fix $x \in \Lambda^\infty$. By definition, we have

$$\sigma^n(1_x) = \sigma^m(1_x) \iff (1_x)(s + m, t + m) = (1_x)(s + n, t + n) \quad \text{for } s \leq t \quad (4.4)$$

$$\iff x(s + m, t + m + 1) = x(s + n, t + n + 1) \quad \text{for } s \leq t$$

Now if $x(s + m, t + m + 1) = x(s + n, t + n + 1)$ for all $s \leq t \in \mathbb{N}^2$, then the uniqueness of factorisations in $\Lambda$ ensures that $x(s + m, t + m) = x(s + n, t + n)$ for all $s \leq t \in \mathbb{N}^2$. Conversely if $x(s + m, t + m) = x(s + n, t + n)$ for all $s \leq t \in \mathbb{N}^2$, then replacing $t$ with $t + 1$ gives $x(s + m, t + m + 1) = x(s + n, t + n + 1)$ for all $s \leq t \in \mathbb{N}^2$. Hence (4.4) shows that

$$\sigma^m(1_x) = \sigma^n(1_x) \iff x(s + m, t + m) = x(s + n, t + n) \quad \text{for } s \leq t \in \mathbb{N}^2$$

$$\iff \sigma^m(x) = \sigma^n(x),$$

establishing the claim. Thus it suffices to show that $M_{1}^{1\Lambda}$ satisfy (H3) if and only if $\Lambda^\infty$ has an aperiodic element.

Suppose first that there exists an aperiodic path $x \in \Lambda^\infty$. Fix $m \in \mathbb{Z}^2$, and write $m = m_+ - m_-$ where $m_+, m_- \in \mathbb{N}^2$. Since $|v(\Lambda^\infty)|w \in \{0, 1\}$ for all $v, w \in \Lambda^0$, $i = 1, 2$, we have that $x$ is completely determined by its restriction to the objects of $\Omega_d$; that is, by the function from $\mathbb{N}^d$ to $\Lambda^i$ given by $n \mapsto x(n)$. Since $x$ is aperiodic, it follows that $\sigma^{m_+}(x(n)) \neq \sigma^{m_-}(x(n))$ for some $n \in \mathbb{N}^2$. But then with $N := n + m_-$, we have $x(N + m_+ - m_-) \neq x(N)$, and $w := x[N + m_+ - m_-] \in W_{1\Lambda}^{N + m_+ - m_-}$ satisfies $w(N) \neq w(N + m)$. Since $m \in \mathbb{Z}^2$ was arbitrary, this establishes that $M_{1}^{1\Lambda}, M_{2}^{1\Lambda}$ satisfy (H3).

Now suppose that $M_{1}^{1\Lambda}, M_{2}^{1\Lambda}$ satisfy (H3). For each $m \in \mathbb{Z}^2 \{0\}$, fix $w_m \in W_{1\Lambda}^m$ and $l_m \in \mathbb{N}^2$ such that $0 \leq l_m, l_m + m \leq S(w_m)$ and $w_m(l_m) \neq w_m(l_m + m)$. Let $\lambda_m$ be the unique path in $\Lambda$ such that $w_m = w_{\lambda_m}^\Lambda$. We will construct an infinite path $x$ which contains infinitely many occurrences of each $\lambda_m$; this will ensure that there is no $m$ for which a sufficiently large shift of $x$ has period $m$, and hence that $x$ is aperiodic. The details of this construction, and the verification that the resulting $x$ is aperiodic constitute the remainder of the proof.

Let $\{m_i : i \in \mathbb{N}\}$ be a listing of $\mathbb{Z}^2 \{0\}$. Fix an arbitrary $v \in \Lambda^0$, and for each $i \in \mathbb{N}$, let $\alpha_i$ be any element of $v(\Lambda^1)r(\lambda_{m_i})$, and let $\beta_i$ be any element of $s(\lambda_{m_i})(\Lambda^1)v$ with the property that $d_2(\alpha_i, \beta_i, \lambda_{m_i}) \geq 1$; this is possible because $\Lambda$ is strongly connected and has no sources.
For $i \in \mathbb{N}$, let $\rho_i = \alpha_i \lambda_i$, and let $\tau_i = \rho_1 \rho_2 \cdots \rho_i$. Let $x$ be the infinite path $x = \tau_1 \tau_2 \tau_3 \cdots$. We claim that $x$ is aperiodic.

To see this, let $s, t \in \mathbb{N}$ be distinct, and let $I_{s,t}$ be the element of $\mathbb{N}$ such that $m_{I_{s,t}} = t - s$. Let $J = \max\{s_1, s_2, t_1, t_2\}$; since $d_1(\rho_i) \geq (1, 1)$, we have that $i \geq J$ implies $d_1(\tau_1 \cdots \tau_i) \geq s, t$. Let $K = \max\{I_{s,t}, J + 1\}$, and define $N = d_1(\tau_1 \cdots \tau_{K-1}) + d_1(\rho_1 \cdots \rho_{I_{s,t}-1}) + d(\alpha_{I_{s,t}}) + l_{t-s} - s$. We have $N \geq 0$ by choice of $K$, and

$$\sigma^s(x)(N) = x(N + s) = x_1 \cdots \tau_{K-1} \rho_1 \cdots \rho_{I_{s,t}-1} \alpha_{I_{s,t}} + l_{t-s}$$

A similar calculation shows that $\sigma^s(x)(N) = \lambda_{m_{I_{s,t}}}(l_{t-s} + (t - s))$, and hence $\sigma^s(x)(N) \neq \sigma^t(x)(N)$ by our choice of $\lambda_{m_{I_{s,t}}}$. It follows that $\sigma^s(x) \neq \sigma^t(x)$, and since $s, t \in \mathbb{N}$ were arbitrary, that $x$ is aperiodic.

Remark 4.4. The preceding proof actually shows: (1) that Proposition 4.3 applies to $M_\lambda^1$ for any $\Lambda$ satisfying the property of Lemma 3.7 for $p = 1$; and (2) that $1\Lambda$ is strongly connected (resp. strongly connected and contains an infinite path) if and only if $\Lambda$ has the same property. Since our motivation is to prove Theorem 4.1, we have compressed this into a single result.

Notation 4.5. Let $\Lambda$ be a finite strongly connected 2-graph with an aperiodic infinite path. We write $A^{1\Lambda}$ for the $C^*$-algebra associated to $M_\lambda^1$ as in [16]. That is, $A^{1\Lambda}$ is the universal $C^*$-algebra generated by a family $\{s_{u,v} : u, v \in W^{1\Lambda}, u(S(u)) = v(S(v))\}$ of partial isometries satisfying

\begin{align*}
(4.5) \quad & s_{u,u} = s_{v,u}^* \quad \text{for } u, v \in W^{1\Lambda}, \\
(4.6) \quad & s_{u,v}s_{v,w} = s_{u,w} \quad \text{for } u, v, w \in W^{1\Lambda}, \\
(4.7) \quad & s_{u,v} = \sum_{w \in W^{1\Lambda}, u(S(u))=w(0)} s_{u,w}s_{v,w}^* \quad \text{for } u, v, w \in W^{1\Lambda}, j \in \{1, 2\}; \text{ and} \\
(4.8) \quad & s_{a,a}s_{b,b} = 0 \quad \text{for distinct } a, b \in W^{1\Lambda}_0.
\end{align*}

Lemma 4.6. Let $(\Lambda, d)$ be a finite strongly connected 2-graph which has an aperiodic infinite path. Then $C^*(\Lambda)$ is isomorphic to $A^{1\Lambda}$.

Proof. The factorisation property ensures that if $\Lambda$ is strongly connected and contains an infinite path, then $\Lambda$ has no sources. By Theorem 3.5, we have that $C^*(\Lambda)$ is isomorphic to $C^*(1\Lambda)$, so it suffices to show that $C^*(1\Lambda)$ is isomorphic to $A^{1\Lambda}$. It is easy to check using Definition 2.4(i)–(iv), relations (4.5)–(4.8), and the universal properties of $A^{1\Lambda}$ and $C^*(1\Lambda)$ that there exists a homomorphism $\pi : A^{1\Lambda} \to C^*(1\Lambda)$ satisfying $\pi(s_{w_{\lambda},w_{\mu}}) = s_{\lambda}s_{\mu}^*$ for all $\lambda, \mu \in 1\Lambda$, and that there exists a homomorphism $\psi : C^*(1\Lambda) \to A^{1\Lambda}$ satisfying $\psi(s_\lambda) = s_{w_{\lambda}1\Lambda}$. Since these two homomorphisms are mutually inverse, the result follows.

Remark 4.7. The argument of statement (2) of Proposition 4.3 shows that if $\Lambda$ has no sources, then for any $q \geq 1$, the coordinate matrices of $q\Lambda$ will satisfy (H2) only if $\Lambda$ is strongly connected and has no sources. In particular, there exists $q \in \mathbb{N}$ such that $M_\lambda^q$ satisfy (H0)–(H3) if and only if $M_\lambda^1$ satisfy (H0)–(H3).
Proof of Theorem 4.1. Theorem 5.9, Proposition 5.11, and Corollary 6.4 of \cite{15} combined with the previous two results show that $C^*(\Lambda)$ is simple, purely infinite and nuclear. We have that $C^*(\Lambda)$ is unital with $1_{C^*(\Lambda)} = \sum_{v \in \Lambda^0} s_v$. Proposition 2.14 of \cite{16} establishes (4.1)–(4.3).

Remarks 4.8. (1) The proof of \cite[Proposition 2.14]{16} does not make any use of relations (H2) and (H3). Hence the formulas for $K_*(C^*(\Lambda))$ in Theorem 4.1 hold when $\Lambda$ is a finite $k$-graph with no sinks or sources, even if it is not strongly connected and does not have an aperiodic infinite path. However, in this case $C^*(\Lambda)$ is not necessarily simple and purely infinite, and so is not determined up to isomorphism by its $K$-theory.

(2) The formulas for $K_*(C^*(\Lambda))$ given in Theorem 4.1 are in terms of the coordinate matrices $M^1_{1}\Lambda$ of the dual $k$-graph. Proposition 5.1 of \cite{6} shows that the same formulas hold if all instances $M^1_{1}\Lambda$ are replaced with $M^1\Lambda$, but it is unclear how to show this directly.

References