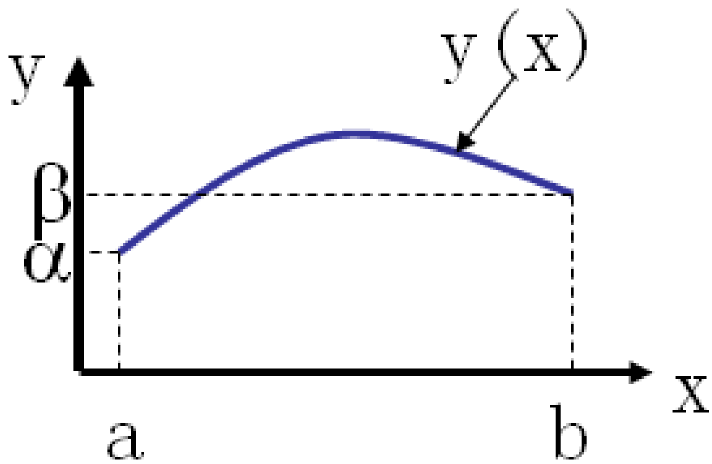
Chapter 12. Ordinary Differential Equation Boundary Value (BV) Problems

In this chapter we will learn how to solve ODE boundary value problem. BV ODE is usually given with x being the independent space variable.

$$y'' + p(x)\, y' + q(x)\; y = f(x) \qquad\qquad a \le x \le b \qquad\qquad (1a)$$

and the boundary conditions (*BC*) are given at both end of the domain e.g. y(a) = α and y(b) = β. They are generally fixed boundary conditions or *Dirichlet* Boundary Condition but can also be subject to other types of BC e.g. *Neumann* BC or *Robin* BC.



14.1 LINEAR FINITE DIFFERENCE (FD) METHOD

Finite difference method converts an ODE problem from calculus problem into algebraic problem. In FD, y′ and y″ are expressed as the difference between adjacent y values, for example,

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} \qquad (1)$$

which are derived from the Taylor series expansion,

$$y(x+h) = y(x) + h\, y'(x) + (h^2/2)\, y''(x) + \ldots \qquad (2)$$

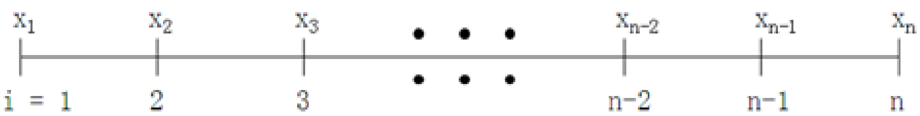
$$y(x\text{-}h) = y(x) - h\, y'(x) + (h^2/2)\, y''(x) + \ldots \qquad (3)$$

If (2) is added to (3) and neglecting the higher order term (O (h^3)), we will get

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \tag{4}$$

The difference Eqs. (1) and (4) can be implemented in [x_1 = a, x_n = b] (see Figure) if few finite points n are defined and dividing domain [a,b] into n-1 intervals of h which is defined

$$h = \frac{x_n - x_1}{n - 1} = x_{i+1} - x_i \tag{5}$$



[x_1,x_n] domain divided into n-1 intervals.

FD method gives derivative of y values at point i, y_i as follow

$$y'(x_i) \approx \frac{y(x_{i+1}) - y(x_{i-1})}{2h} \tag{6}$$

$$y''(x_i) \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} \tag{7}$$

Substituting (6) & (7) into (1a), we get

$$\left[1 - \frac{1}{2} h\, p(x_i)\right] y(x_{i-1}) - \left[2 - h^2 q(x_i)\right] y(x_i) + \left[1 + \frac{1}{2} h\, p(x_i)\right] y(x_{i+1}) = h^2\, f(x_i) \tag{8}$$

or can be simplified to,

$$\left[1 - \frac{1}{2} h\, p(x_i)\right] y_{i-1} - \left[2 - h^2 q(x_i)\right] y_i + \left[1 + \frac{1}{2} h\, p(x_i)\right] y_{i+1} = h^2\, f(x_i) \tag{9}$$

Eq. (9) is then applied to each of the internal nodes, i = 2, … ,n-1. This will produce a system of linear equations of tri-diagonal form. The system of linear equations can then be solved using the Thomas algorithm (but we will solve using sparse matrix technique).

----- Example 1 ---

Solve the following 2nd order ODE,

$$y'' + 7y' + 3y = 1$$

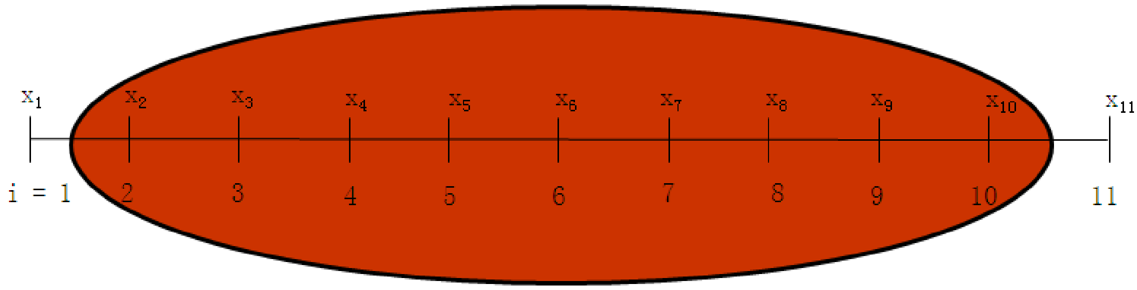
for [0,1] with ***Dirichlet boundary condition***, y(0) = 0 and y(1) = 1, using FD with h = 0.1 (Note in this example $p(x) = 7$, $q(x) = 3$ and $f(x) = 1$). The ***difference equation*** (9) for this problem is,

$$(1 - 3.5h)y_{i-1} - (2 - 3h^2)y_i + (1 + 3.5h)y_{i+1} = h^2$$

for h = 0.1,

$$0.65\ y_{i-1} - 1.97\ y_i + 1.35\ y_{i+1} = 0.01 \qquad \text{...(C-1)}$$

The difference equation is applied to the internal nodes (i = 2 - 10),



Applying (C-1) at i = 2 – 10, we obtain the following system of linear equations

$$
\begin{array}{lllllllllll}
i=2\rightarrow & -1.97y_2 & +1.35y_3 & & & & & & & & = 0.01 - 0.65y_1 \\
i=3\rightarrow & 0.65y_2 & -1.97y_3 & +1.35y_4 & & & & & & & = 0.01 \\
i=4\rightarrow & & 0.65y_3 & -1.97y_4 & +1.35y_5 & & & & & & = 0.01 \\
i=5\rightarrow & & & 0.65y_4 & -1.97y_5 & +1.35y_6 & & & & & = 0.01 \\
i=6\rightarrow & & & & 0.65y_5 & -1.97y_6 & +1.35y_7 & & & & = 0.01 \\
i=7\rightarrow & & & & & 0.65y_6 & -1.97y_7 & +1.35y_8 & & & = 0.01 \\
i=8\rightarrow & & & & & & 0.65y_7 & -1.97y_8 & +1.35y_9 & & = 0.01 \\
i=9\rightarrow & & & & & & & 0.65y_8 & -1.97y_9 & +1.35y_{10} & = 0.01 \\
i=10\rightarrow & & & & & & & & 0.65y_9 & -1.97y_{10} & = 0.01 - 1.35y_{11}
\end{array}
$$

Note that nodes i = 1 & 11 are ***end nodes,*** which have specified values and they do not appear in the system. The above system of linear equations can be written in tri-diagonal matrix form, which can be solved using ***Thomas algorithm*** but we will solve using sparse technique.

$$\begin{bmatrix} -1.97 & 1.35 & 0 & \cdots & & & & & 0 \\ 0.65 & -1.97 & 1.35 & 0 & \cdots & & & & 0 \\ 0 & 0.65 & -1.97 & 1.35 & 0 & \cdots & & & 0 \\ 0 & 0 & 0.65 & -1.97 & 1.35 & 0 & \cdots & & 0 \\ 0 & \cdots & 0 & 0.65 & -1.97 & 1.35 & 0 & \cdots & 0 \\ 0 & & \cdots & 0 & 0.65 & -1.97 & 1.35 & 0 & 0 \\ 0 & & & \cdots & 0 & 0.65 & -1.97 & 1.35 & 0 \\ 0 & & & & \cdots & 0 & 0.65 & -1.97 & 1.35 \\ 0 & & & & & \cdots & 0 & 0.65 & -1.97 \end{bmatrix} \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ -1.34 \end{pmatrix}$$

The solution of the above system is

$\mathbf{y}^T = [y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}]$

= [0.6420, 0.9443, 1.0762, 1.1233, 1.1284, 1.1131, 1.0885, 1.0598, 1.0299]

Note that the difference equations at nodes i = 2 and i = n-1 = 10 are slightly modified by the end nodes,

$$i = 2 \quad -\left[2 - h^2 q(x_2)\right] y_2 + \left[1 + \frac{1}{2} h\, p(x_2)\right] y_3 = h^2 f(x_2) - \left[1 - \frac{1}{2} h\, p(x_2)\right] y_1 \qquad (10)$$

$$i = 10 \quad \left[1 - \frac{1}{2} h\, p(x_9)\right] y_9 - \left[2 - h^2 q(x_{10})\right] y_{10} = h^2 f(x_{10}) - \left[1 + \frac{1}{2} h\, p(x_{10})\right] y_{11} \qquad (11)$$

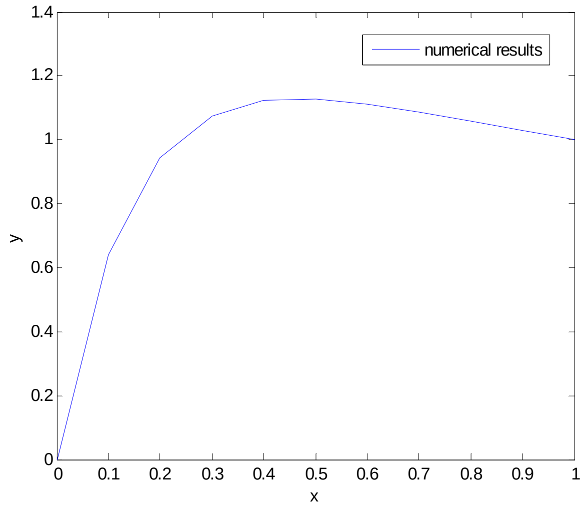
The sample MATLAB code (*Example1.m*) I developed to solve this problem is shown below.

```
%***************************************************************************
% Program to solve linear ODE boundary value problems with FD
% Both ends are subjected to Dirichlet boundary conditions
% inputs :
%     a,b  are starting and ending points
%     alfa,beta are the y values at the starting and ending points
%     n  is the number of interval
```

```
% outputs :
%     y are the values of y at node points
%*****************************************************************************
clear;clf;

%********** INPUT ******************************************************
a = input('Enter the starting x >');
b = input('Enter the ending x >');
alfa = input('Enter the y value at starting x >');
beta = input('Enter the y value at ending x >');
n = input('Enter the number of divisions >');
pfunc = input('Enter the function p(x) please: ','s');
qfunc = input('Enter the function q(x) please: ','s');
ffunc = input('Enter the function f(x) please: ','s');
p = inline(pfunc);
q = inline(qfunc);
f = inline(ffunc);

%*********** Constants ************
h = (b - a)/n;                    % the interval size
x = linspace(a,b,n+1);

%************ Elements of the matrix A
A(1:n-1,1:n-1) = 0.0;             % initialize matrix A

for i = 1:n-1                     % this loop calculate element for nodes 2 to n
                                  % in the diagram but in the
                                  % matrix it is from i = 1 to n-1
   if i == 1
      A(i,i)= -(2. - q(x(i+1)) * h*h);
      A(i,i+1) = 1 + 0.5 * p(x(i+1)) * h;
      ff(i) = f(x(i+1)) * h*h - (1. - p(x(i+1)) * 0.5 * h) * alfa;
   elseif i == n-1
      A(i,i-1) = 1. - p(x(i+1)) * 0.5 * h;
      A(i,i) = -(2. - q(x(i+1)) * h*h);
      ff(i) = f(x(i+1)) * h*h - (1 + 0.5 * p(x(i+1)) * h) * beta;
   else
      A(i,i-1) = 1. - p(x(i+1)) * 0.5 * h;
      A(i,i) = -(2. - q(x(i+1)) * h*h);
      A(i,i+1) = 1 + 0.5 * p(x(i+1)) * h;
```

```
      ff(i) = f(x(i+1)) * h*h;
   end
end

%************ Solve the system of linear equations using sparse matrix
ASparse = sparse(A);
y = ASparse\ff';

%*********** PLOT OF RESULT ********
plot(x,[alfa;y;beta],'b-')
xlabel('x'),ylabel('y');
legend('numerical results')
```

```
Enter the starting x >0
Enter the ending x >1
Enter the y value at starting x >0
Enter the y value at ending x >1
Enter the number of divisions >10
Enter the function file of p(x) please: 7.
Enter the function file of q(x) please: 3.
Enter the function file of f(x) please: 1.
```



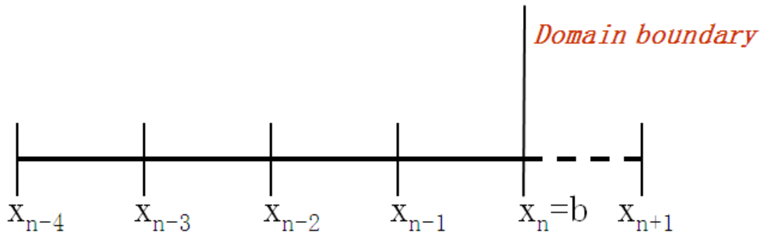
Now suppose the ODE Eq. (1)

$$y'' + p(x)\, y' + q(x)\; y = f(x) \qquad\qquad a \le x \le b \tag{1}$$

is now subject to boundary conditions y(a) = α and y′(b) = β. The boundary at b is a derivative boundary condition or *Neumann* Boundary Condition. For the problem the value of y(b) must be calculated (*part of the solution*) so difference equations must be written for i = 1,2,…,n. For i = 1,2, 1,2,…,n-1 according to the FD equation (9) is but for i = n it is different as y′′(b) needs $y(x_n+h)$ as shown below.

$$y''(x_n) = \frac{y(x_n + h) - 2y(x_n) + y(x_n - h)}{h^2} = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \tag{12}$$

It is clear from (12) that y(b+h) is needed, and this is outside the domain as shown in the figure below.



To calculate y_{n+1}, we will make use of the given $y'(x_n) = \beta$ and following (6),

$$y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h} \tag{13}$$

So,

$$y_{n+1} = y_{n-1} + 2hy'_n \tag{14}$$

If (14) is substituted into (12)

$$y''_n = \frac{2hy'_n - 2y_n + 2y_{n-1}}{h^2} \tag{15}$$

and substituting (15) and $y'_n = \beta$ into (1), we get

$$2y_{n-1} - \left(2 - h^2 q(x_n)\right) y_n = h^2 f(x_n) - 2h\beta - h^2 p(x_n)\beta \tag{16}$$

Eq. (16) is the FD equation applied at node i = n.

----- **Example 2** ---

Solve the following 2nd order ODE,

$$y'' + 5y' + 4y = 1$$

For [0,1] and boundary conditions y(0) = 1 and $y'(1) = 0$. Use h = 0.1, in this problem *p(x) = 5*, *q(x) = 4* and *f(x) = 1.0*. The FD equations are given by (9),

$$(1 - 2.5h)y_{i-1} - (2 - 4h^2)y_i + (1 + 2.5h)y_{i+1} = h^2$$

For h = 0.1,

$$0.75\ y_{i-1} - 1.96\ y_i + 1.25\ y_{i+1} = 0.01 \qquad \text{...(C-1)}$$

(C-1) is applicable for i = 2 – 10, but for i = 11, (16) is used

$$2y_9 - 1.96y_{10} = 0.01 \qquad \text{...(C-2)}$$

The system of linear equations to be solved is

$$\begin{array}{lllllllllllr}
i = 2 \rightarrow & -1.96y_2 & 1.25y_3 & & & & & & & & & = 0.01 - 0.75y_1 \\
i = 3 \rightarrow & 0.75y_2 & -1.96y_3 & 1.25y_4 & & & & & & & & = 0.01 \\
i = 4 \rightarrow & & 0.75y_3 & -1.96y_4 & 1.25y_5 & & & & & & & = 0.01 \\
i = 5 \rightarrow & & & 0.75y_4 & -1.96y_5 & 1.25y_6 & & & & & & = 0.01 \\
i = 6 \rightarrow & & & & 0.75y_5 & -1.96y_6 & 1.25y_7 & & & & & = 0.01 \\
i = 7 \rightarrow & & & & & 0.75y_6 & -1.96y_7 & 1.25y_8 & & & & = 0.01 \\
i = 8 \rightarrow & & & & & & 0.75y_7 & -1.96y_8 & 1.25y_9 & & & = 0.01 \\
i = 9 \rightarrow & & & & & & & 0.75y_8 & -1.96y_9 & 1.25y_{10} & & = 0.01 \\
i = 10 \rightarrow & & & & & & & & 0.75y_9 & -1.96y_{10} & 1.25y_{11} & = 0.01 \\
i = 11 \rightarrow & & & & & & & & & 2y_{10} & -1.96y_{11} & = 0.01
\end{array}$$

In matrix form,

$$\begin{bmatrix} -1.96 & 1.25 & 0 & \cdots & & & & & & 0 \\ 0.75 & -1.96 & 1.25 & 0 & \cdots & & & & & 0 \\ 0 & 0.75 & -1.96 & 1.25 & 0 & \cdots & & & & 0 \\ 0 & \cdots & 0.75 & -1.96 & 1.25 & 0 & \cdots & & & 0 \\ 0 & & \cdots & 0.75 & -1.96 & 1.25 & 0 & \cdots & & 0 \\ 0 & & & \cdots & 0.75 & -1.96 & 1.25 & 0 & \cdots & 0 \\ 0 & & & & \cdots & 0.75 & -1.96 & 1.25 & 0 & 0 \\ 0 & & & & & \cdots & 0.75 & -1.96 & 1.25 & 0 \\ 0 & & & & & & \cdots & 0.75 & -1.96 & 1.25 \\ 0 & & & & & & & \cdots & 2 & -1.96 \end{bmatrix} \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \end{pmatrix} = \begin{pmatrix} -0.74 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}$$

The solution vector is

y^T = [0.7051,0.5136,0.3903,0.3118,0.2628,0.2329,0.2155,0.2062,0.2020,0.2011]

As a practice, modify the code used in Example 1 to solve this one.

Another possible boundary condition is *Robin* Boundary Condition. In this case the BC is expressed as an expression,

$$A_2\, y_n + B_2\, y'_n = \beta \tag{17}$$

Using backward difference to express y'_n,

$$A_2\, y_n + B_2 \left(\frac{y_n - y_{n-1}}{h} \right) = \beta \tag{18}$$

Or,

$$y_n = \frac{\beta h + B_2 y_{n-1}}{(A_2 h + B_2)} \tag{19}$$

----- Example 3 ---

Solve the following 2nd order ODE,

$$y'' + 5y' + 4y = 1$$

For [0,1] and boundary conditions: y(0)=0 and y(1)–0.5y'(1)=0.5. Use FD method with h = 0.1. For i = 2 – 10 the FD equations are the same as previous example, but for i = 11 the FD equation is calculated (19) with $A_2 = 1$, $B_2 = -0.5$ dan $\beta = 0.5$.

i = 11 → $-1.25\, y_{10} + y_{11} = -0.125$

The formed system of linear equations is

$$
\begin{bmatrix}
-1.96 & 1.25 & 0 & \cdots & & & & & & 0 \\
0.75 & -1.96 & 1.25 & 0 & \cdots & & & & & 0 \\
0 & 0.75 & -1.96 & 1.25 & 0 & \cdots & & & & 0 \\
0 & \cdots & 0.75 & -1.96 & 1.25 & 0 & \cdots & & & 0 \\
0 & & \cdots & 0.75 & -1.96 & 1.25 & 0 & \cdots & & 0 \\
0 & & & \cdots & 0.75 & -1.96 & 1.25 & 0 & \cdots & 0 \\
0 & & & & \cdots & 0.75 & -1.96 & 1.25 & 0 & 0 \\
0 & & & & & \cdots & 0.75 & -1.96 & 1.25 & 0 \\
0 & & & & & & \cdots & 0.75 & -1.96 & 1.25 \\
0 & & & & & & & \cdots & -1.25 & 1
\end{bmatrix}
\begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \end{pmatrix}
=
\begin{pmatrix} 0.0 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ -0.125 \end{pmatrix}
$$

The solution is

$\mathbf{y}^T$ = [0.2198, 0.3446, 0.4165, 0.4543, 0.4704, 0.473, 0.4675, 0.4572, 0.4444, 0.4305]

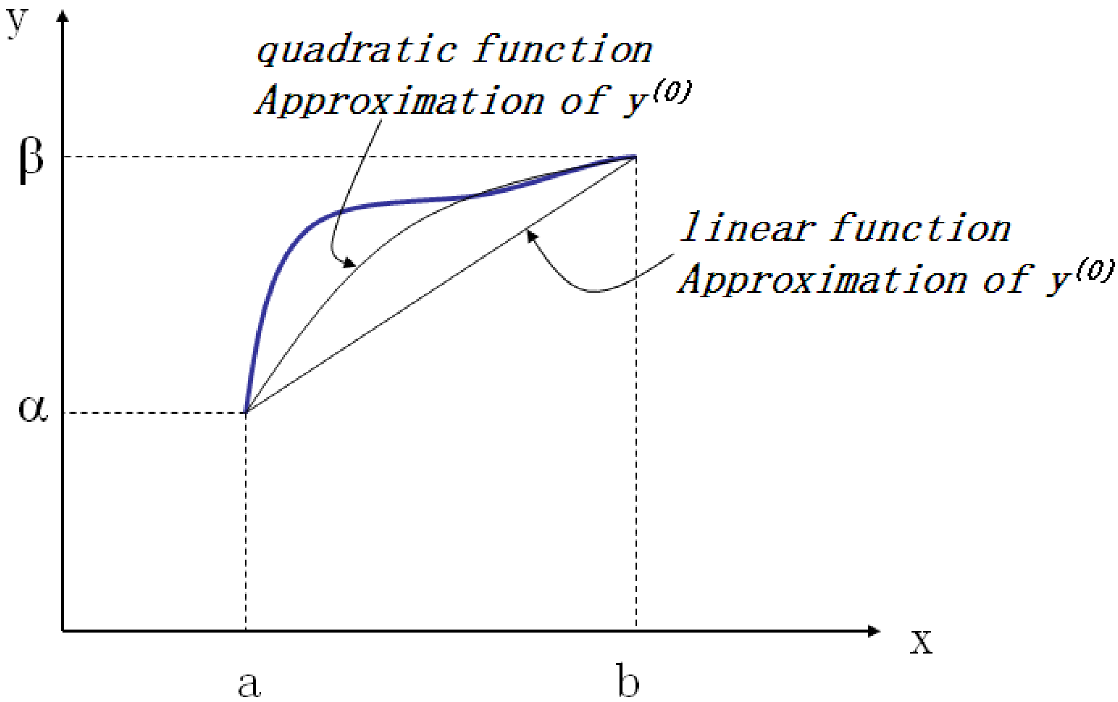
As a practice, modify the code used in Example 1 to solve this one.

14.2 NON-LINEAR FINITE DIFFERENCE (FD) METHOD

The solution of non-linear ODE using FD method is similar to the linear FD except for non-linear ODE the solutions is obtained iteratively. In non-linear problem *p(x,y)* and *q(x,y)* are function of x and y or derivatives of y, so at each iteration k^{th} , *p(x,y)* and *q(x,y)* are calculated using the y values at $(k\text{-}1)^{th}$ iteration. In other words we linearize the ODE into,

$$y''^{(k)} + p(x, y^{(k-1)})\, y'^{(k)} + q(x, y^{(k-1)})\, y^{(k)} = f(x) \qquad (20)$$

As in any iterative techniques, we must start from an initial estimate $y^{(0)}$. The values of the initial estimate can affect the rate of convergence, so we must prudently guess $y^{(0)}$. The simplest technique is to assume linear y(x) as shown in the following figure.



Then iterations are done until the left side differ within a certain tolerance from the right side. The above method involves solving linear FD equations again and again. A more explicit technique can also be obtained, we start by writing the ODE again,

$$y'' = f(x) - p(x,y)y' - q(x,y)y \tag{21}$$

After expressing y′ and y′′ using (6) and (7) and substituting them into (21),

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f(x_i) - p(x_i, y_i)\frac{y_{i+1} - y_{i-1}}{2h} - q(x_i, y_i)y_i \tag{22}$$

Rearranging (22), y_i can be explicitly expressed as

$$y_i = \frac{y_{i+1} + y_{i-1} - h^2 f(x_i) + h\, p(x_i, y_i)\frac{y_{i+1} - y_{i-1}}{2} + h^2 q(x_i, y_i)y_i}{2} \tag{23}$$

Note that y_i can be calculated iteratively if all ys on the right hand side are the known from the previous calculation or initially assumed at the beginning of the iteration.

$$y_i^{(k+1)} = \frac{1}{4}\left[2y_{i+1}^{(k)} + 2y_{i-1}^{(k)} - 2h^2\, f(x_i) + h\, p(x_i, y_i^{(k)})(y_{i+1}^{(k)} - y_{i-1}^{(k)}) + 2h^2 q(x_i, y_i^{(k)})\, y_i^{(k)}\right] \quad (24)$$

Fausett suggested that convergence can be accelerated if we add ωy_i to both side of (23),

$$y_i = \frac{1}{4(1+\omega)}\left[2y_{i+1} + 4\omega y_i + 2y_{i-1} - 2h^2\, f(x_i) + h\, p(x_i, y_i)(y_{i+1} - y_{i-1}) + 2h^2 q(x_i, y_i)\, y_i\right] \quad (25)$$

And y_i is calculated iteratively,

$$y_i^{(k+1)} = \frac{1}{4(1+\omega)}\left[2y_{i+1}^{(k)} + 4\omega y_i^{(k)} + 2y_{i-1}^{(k)} - 2h^2\, f(x_i) + h p(x_i, y_i^{(k)})(y_{i+1}^{(k)} - y_{i-1}^{(k)}) + 2h^2 q(x_i, y_i^{(k)})\, y_i^{(k)}\right] \quad (26)$$

The iteration process is stopped when a specified convergence criteria has been reached. An acceptable convergence criteria is

$$\left\| \mathbf{y}_i^{k+1} - \mathbf{y}_i^{k} \right\|_\infty < \varepsilon \quad (27)$$

----- Example 4 --

Solve the following non-linear ODE,

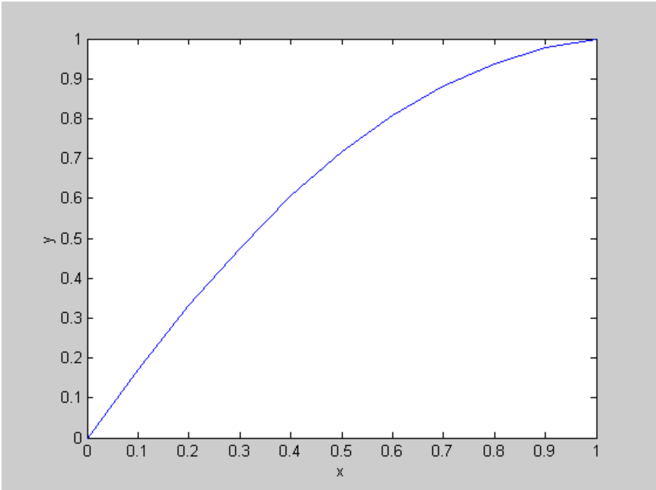
$$y'' + (1+y)y' + (1+y)y = 1$$

For [0,1] and boundary conditions y(0) = 0 and y(1) = 1. *p(x,y)* = *q(x,y)* = 1 + y and *f(x)* = 1. Use the explicit method with h = 0.1 and initial estimate $\mathbf{y}_i^{(0)}$ for i = 2-10 which assumed linear $\mathbf{y}^{(0)}$ = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]. The explicit form of y (Eq. 26) for this example is,

$$y_i^{(k+1)} = \frac{1}{4(1+\omega)}\left[2y_{i+1}^{(k)} + 4\omega y_i^{(k)} + 2y_{i-1}^{(k)} - 2h^2 + h(1+y_i^{(k)})(y_{i+1}^{(k)} - y_{i-1}^{(k)}) + 2h^2(1+y_i^{(k)})y_i^{(k)}\right]$$

$$= \frac{1}{4(1+\omega)}\left[2y_{i+1}^{(k)} + 4\omega y_i^{(k)} + 2y_{i-1}^{(k)} - 0.02 + 0.1(1+y_i^{(k)})(y_{i+1}^{(k)} - y_{i-1}^{(k)}) + 0.02(1+y_i^{(k)})y_i^{(k)}\right]$$

The result is shown in Figure below.



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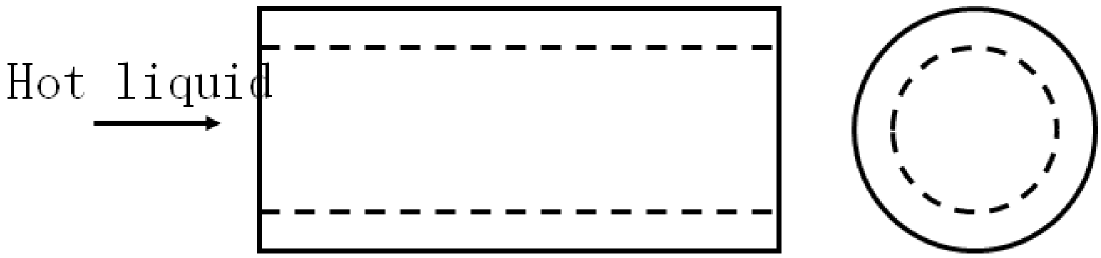
14. 3 Tutorial Questions

1. Find the temperature profile inside the tube wall. Hot fluid flows inside the tube such that the inside temperature is 100^{0}C. The differential equation for the temperature distribution is given by

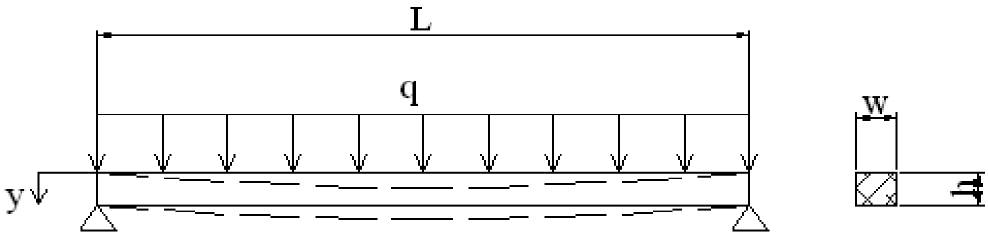
$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}=0$$

The boundary conditions are T(100) = 100^{0}C and T(150) = 0^{0}C. This is a linear 2nd order linear ODE with Dirichlet boundary conditions, so use linear FD technique with h = 10.



Hot fluid flowing inside a tube.

2. Deflection of prismatic simply supported beam shown below



is given by the following equation.

$$EI\frac{d^2y}{dx^2} = -\frac{qLx}{2} + \frac{qx^2}{2}$$

Where V is uniformly distributed load, L is the length of the beam, I is the moment of inertia of the beam cross section ($I = wh^3/12$, w is width and h is height of the beam) and E is the modulus of elasticity of the beam. For the case in hand E = 10GPa, L = 2.0m, w = 5.0cm, h = 10.0cm, V = 1500N/m and $I = 4.166x10^{-6}$ m^4, Find y(x).

It is clear from the problem that the boundary conditions are *Dirichlet* i.e., y (0) = y(2) = 0., Rewrite the governing equation

$$y'' = \frac{Vx}{2EI}(x - L) = 0.018x(x - 2)$$

From the above, *p(x)* = 0, *q(x)* = 0 and *f(x)* = 0.018 x (x – 2.). Divide the beam into 10 interval giving h = 0.2 and generate plot of the beam deflection.

3. Use linear shooting method and linear Finite Difference Method to find the solution of

ODE	Range	Boundary conditions	Analytical solution	h
$y'' = 4(y - x)$	[0,1]	$y(0) = 0$ $y(1) = 2$	$y(x) = \dfrac{e^2}{e^4 - 1}(e^{2x} - e^{-2x}) + x$	0.25

4. Use FD method (*Neumann Boundary conditions*) to find the solution of

ODE	Range	Boundary conditions	h
$y'' = -5y' - 4y + 1$	[0,1]	$y(0) = 1$ $y'(1) = 0$	0.25

5. Use FD method (Robin Boundary conditions) to find the solution of

ODE	Range	Boundary conditions	h
$y'' = -4y' - 6.25y + e^x$	[0,1]	$y(0) = 1$ $y(1) - 0.5\,y'(1) = 0.5$	0.25

6. Use non-linear FD method to find the solution of

ODE	Range	Boundary conditions	Analytical solution	h
$y'' = y^3 - y\,y'$	[1,2]	$y(1) = \frac{1}{2}$ $y(2) = \frac{1}{3}$	$y(x) = \dfrac{1}{x+1}$	0.25

Comment on the use of different ω say from 0.5 – 1.5. Adopt the following convergence criteria

$$\left\| \mathbf{y}_i^{k+1} - \mathbf{y}_i^{k} \right\|_\infty < 10^{-5}$$