THE PRIMITIVE IDEALS OF THE CUNTZ-KRIEGER ALGEBRA OF A ROW-FINITE HIGHER-RANK GRAPH WITH NO SOURCES

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Abstract. We catalogue the primitive ideals of the Cuntz-Krieger algebra of a row-finite higher-rank graph with no sources. Each maximal tail in the vertex set has an abelian periodicity group of finite rank at most that of the graph; the primitive ideals in the Cuntz-Krieger algebra are indexed by pairs consisting of a maximal tail and a character of its periodicity group. The Cuntz-Krieger algebra is primitive if and only if the whole vertex set is a maximal tail and the graph is aperiodic.

1. Introduction

Graph \( C^* \)-algebras were introduced in the early 1980’s by Enomoto and Watatani [6] as an alternative description of the Cuntz-Krieger algebras invented in [4]. Enomoto and Watatani considered only finite graphs, but since the late 1990’s substantial work has gone into describing and understanding the analogous construction for infinite directed graphs in various levels of generality (to name just a few, [2, 8, 10, 14, 18, 23]). A gem in this program is Hong and Szymański’s description [9] of the primitive ideal space of the \( C^* \)-algebra of a directed graph. Hong and Szymański catalogue the primitive ideals of \( C^*(E) \) in terms of elementary structural features of \( E \). They also describe the closure operation in the hull-kernel topology in terms of this catalogue.

In 1999, Robertson and Steger discovered a class of higher-rank Cuntz-Krieger algebras arising from \( \mathbb{Z}^k \) actions on buildings [21]. Shortly afterwards, Kumjian and Pask introduced higher-rank graphs and the associated \( C^* \)-algebras [12] as a simultaneous generalisation of the graph \( C^* \)-algebras of [13] and Robertson and Steger’s higher-rank Cuntz-Krieger algebras. Kumjian and Pask’s \( k \)-graph algebras have since attracted a fair bit of attention. But their structure is more subtle and less well understood than that of graph \( C^* \)-algebras. In particular, while large portions of the gauge-invariant theory of graph \( C^* \)-algebras can be generalised readily to \( k \)-graphs, higher-rank analogues of other structure theorems for graph \( C^* \)-algebras are largely still elusive.

Our main result here is a complete catalogue of the primitive ideals in the \( C^* \)-algebra of a row-finite \( k \)-graph with no sources. Our methods are very different from Hong and Szymański’s, and require a different set of technical tools. We have been unable to describe the hull-kernel topology on \( \text{Prim}(C^*(\Lambda)) \), and we leave this question open for the present.

We begin in Section 2 by introducing \( P \)-graphs \( \Gamma \) and their \( C^* \)-algebras \( C^*(\Gamma) \), where \( P \) is the image of \( \mathbb{N}^k \) under a homomorphism \( f \) of \( \mathbb{Z}^k \). We prove a gauge-invariant uniqueness theorem (Proposition 2.7) and a Cuntz-Krieger uniqueness theorem (Corollary 2.8) for these \( C^* \)-algebras. In Section 3 we consider the primitive ideal space of the pullback \( k \)-graph arising from a \( P \)-graph. We first show that if \( f \) and \( P \) are as above, then the
pullback $f^* \Gamma$ of a $P$-graph $\Gamma$ over $f : \mathbb{N}^k \to P$ and $d : \Gamma \to P$ is a $k$-graph. We then prove in Theorem 3.3 that the irreducible representations of $C^*(f^* \Gamma)$ are in bijection with pairs $(\pi, \chi)$ where $\pi$ is an irreducible representation of $C^*(\Gamma)$, and $\chi$ is a character of ker $f$.

In Section 2, we study maximal tails in $k$-graphs. Building on an idea from [5], we show that each maximal tail $T$ has a well-defined \textit{periodicity group} $\text{Per}(T)$, and contains a large hereditary subset $H$ such that the subgraph $HAT$ consisting of paths whose range and source both belong to $H$ is isomorphic to a pullback of an $(\mathbb{N}^k / \text{Per}(T))$-graph. The algebra $C^*(HAT)$ can be identified with a corner in $C^*(\Lambda T)$. Combining this with our earlier results, we describe a bijection $(T, \chi) \mapsto I_{T, \chi}$ from pairs $(T, \chi)$ where $T$ is a maximal tail of $\Lambda$ and $\chi$ is a character $\text{Per}(T)$ to primitive ideals of $C^*(\Lambda)$. We conclude by showing that $C^*(\Lambda)$ is primitive if and only if $\Lambda^0$ is a maximal tail and $\Lambda$ is aperiodic.

As usual in our subject, our convention is that in the context of $C^*$-algebras, “homomorphism” always means “$*$-homomorphism,” and “ideal” always means “closed two-sided ideal”.

2. $P$-Graphs

We introduce $P$-graphs over finitely generated cancellative abelian monoids $P$, and an associated class of $C^*$-algebras. Our treatment is very brief since these objects are introduced as a technical tool, and the ideas are essentially identical to those developed in [12]. Specialising to $P = \mathbb{N}^k$ in this section also serves to introduce the notation used later for $k$-graphs.

Definition 2.1. Let $P$ be a finitely generated cancellative abelian monoid, which we also regard as a category with one object. A $P$-graph is a countable small category $\Gamma$ equipped with a functor $d : \Gamma \to P$ which has the factorisation property: whenever $\xi \in \Gamma$ satisfies $d(\xi) = p + q$, there exist unique elements $\eta, \zeta \in \Gamma$ such that $d(\eta) = p$, $d(\zeta) = q$ and $\eta = \eta \zeta$.

Observe that if $P = \mathbb{N}^k$, then a $P$-graph is precisely a $k$-graph in the sense of [12].

If $\Gamma$ is a $P$-graph and $p \in P$, then we write $\Gamma^p$ for $d^{-1}(p)$. When $\xi$ is a morphism of $\Gamma$, we write $s(\xi)$ for the domain of $\xi$ and $r(\xi)$ for the codomain of $\xi$. If $d(\xi) = 0$, then the factorisation property combined with the identities $\xi = \text{id}_{r(\xi)}$ and $\xi = \text{id}_{s(\xi)}$ imply that $\xi = \text{id}_{r(\xi)} = \text{id}_{s(\xi)}$. Thus $\Gamma^0 = \{ \text{id}_o : o \text{ is an object of } \Gamma \}$, and we can regard $r, s$ as maps from $\Gamma$ to $\Gamma^0$; we then have $r(\xi) = \xi = s(\xi)$ for all $\xi \in \Gamma$.

Given $\xi, \eta \in \Gamma$ and a subset $E \subseteq \Gamma$, we define

$$\xi E = \{ \xi \zeta : \zeta \in E, r(\zeta) = s(\xi) \}, \quad E \eta = \{ \eta \zeta : \zeta \in E, s(\zeta) = r(\eta) \}, \quad \xi E \eta = \xi E \cap E \eta.$$  

For $X, E, Y \subseteq \Gamma$, we write $X E Y$ for $\bigcup_{\xi \in X, \eta \in Y} \xi E \eta$. We say that the $P$-graph $\Gamma$ is \textit{row-finite with no sources} if $0 < |v \Gamma^p| < \infty$ for all $v \in \Gamma^0$ and $p \in P$.

Example 2.2. Let $P$ be a finitely generated cancellative abelian monoid. Let $\Omega_P := \{(p, q) \in P \times P : \text{ there exists } r \in P \text{ such that } p + r = q\}$.

Since $P$ is cancellative, we may embed it in its Grothendieck group $G$ so that it makes sense to write $p - q$ for $p, q \in P$. If $(p, q) \in \Omega_P$, then $q - p \in P$. Define $d : \Omega_P \to P$ by $d(p, q) = q - p$. Define $r, s : \Omega_P \to P$ by $r(p, q) = p$ and $s(p, q) = q$, and for $(p, q), (q, r) \in \Omega_P$, define $(p, q)(q, r) = (p, r)$; this $(p, r)$ belongs to $\Omega_P$ because $r = p + ((q - p) + (r - q))$.

Then $\Omega_P$ is a category with objects $P$ and identity morphisms $(p, p)$, and $d$ is a functor from $\Omega_P$ to $P$. Cancellativity of $P$ implies that $\Omega_P$ is a $P$-graph. We identify $\Omega_P^0$ with $P$ via $(p, p) \mapsto p$. For $v \in \Omega_P^0$ and $p \in P$, we then have $v \Omega_P^p = \{(v, v + p)\}$, so $\Omega_P$ is row-finite with no sources.
A $P$-graph morphism is a functor $x : \Gamma \to \Sigma$ between $P$-graphs $\Gamma$ and $\Sigma$ such that $d_\Sigma(x(\xi)) = d_\Gamma(\xi)$ for all $\xi \in \Gamma$. For a $P$-graph $\Gamma$, we define

$$\Gamma^\Omega := \{ x : \Omega_P \to \Gamma \mid x \text{ is a } P\text{-graph morphism} \}.$$ 

**Lemma 2.3.** Let $P$ be a finitely generated cancellative abelian monoid and let $\Gamma$ be a row-finite $P$-graph with no sources. For each $v \in \Gamma^0$ there exists $x \in \Gamma^\Omega$ such that $x(0) = v$.

**Proof.** Fix generators $a_1, \ldots, a_k$ for $P$. Let $1 = \sum_{i=1}^k a_i$. Let $v_0 := v$. Recursively for $n \geq 1$ choose $\lambda_n \in \psi_{v_{n-1}\Lambda}$ and set $\psi_n := s(\lambda_n)$. Suppose that $(p, q) \in \Omega_P$. Express $q = \sum_{i=1}^k q_ia_i$ where each $q_i \in \mathbb{N}$. Let $n = \max_{i \leq k} q_i$. Then $d(\lambda_1 \ldots \lambda_n) = p + (q - p) + \sum_{i=1}^k (n - q_i)a_i$. Two applications of the factorisation property give $\lambda_1 \ldots \lambda_n = \xi \eta \zeta$ where $d(\xi) = p$ and $d(\eta) = q - p$. Define $x(p, q) := \eta$. The factorisation property guarantees that there is a well-defined function $x : \Omega_P \to \Gamma$ defined by this formula, and that $x$ is a $P$-graph morphism. We have $x(0) = v$ by construction. \hfill \Box

We associate to each row-finite $P$-graph $\Gamma$ with no sources a $C^*$-algebra $C^*(\Gamma)$. As we shall see in section 3 these $C^*$-algebras are isomorphic to quotients by primitive ideals of crucial building blocks of $k$-graph $C^*$-algebras.

**Definition 2.4.** Let $P$ be a finitely generated cancellative abelian monoid and let $\Gamma$ be a row-finite $P$-graph with no sources. A Cuntz-Krieger $\Gamma$-family in a $C^*$-algebra $B$ is a collection $\{ t_\xi : \xi \in \Gamma \} \subseteq B$ such that

1. (CK1) $\{ t_\xi : v \in \Gamma^0 \}$ is a set of mutually orthogonal projections;
2. (CK2) $t_\xi t_\eta = t_{\xi \eta}$ whenever $s(\xi) = r(\eta)$;
3. (CK3) $t_\xi^* t_\xi = t_{s(\xi)}$ for all $\xi \in \Gamma$; and
4. (CK4) $t_\xi = \sum_{\xi \in \Lambda^0} t_\xi^* t_\xi$ for all $v \in \Lambda^0$ and $p \in P$.

Fix a Cuntz-Krieger $\Gamma$-family $\{ t_\xi : \xi \in \Gamma \}$. For $\xi, \eta \in \Gamma$,

$$t_\xi^* t_\eta = \sum_{\zeta \in r(\eta)\Gamma^d(\xi) + d(\eta)} t_\zeta^* t_\eta.$$ 

Fix $\zeta \in r(\eta)\Gamma^d(\xi) + d(\eta)$. By the factorisation property, there are unique factorisations $\zeta = \theta \theta' = \omega \omega'$ with $d(\theta) = d(\xi)$ and $d(\omega) = d(\eta)$. Relations (CK1) and (CK3) show that $t_\xi^* t_\zeta = 0$ unless $\theta = \xi$, and $t_\xi^* t_\eta = 0$ unless $\omega = \eta$, and we deduce that

$$t_\xi^* t_\eta = \sum_{\xi \xi' = \eta \eta' \in r(\eta)\Gamma^d(\xi) + d(\eta)} t_{\xi'}^* t_\eta.'$$

Hence $C^*(\{ t_\xi : \xi \in \Gamma \}) = \text{span} \{ t_\xi^* t_\eta : \xi, \eta \in \Gamma \}$.

A standard argument now shows that there is a universal $C^*$-algebra $C^*(\Gamma)$ generated by a Cuntz-Krieger family $\{ s_\xi : \xi \in \Gamma \}$. If $P = \mathbb{N}^k$ then the relations of Definition 2.4 are those of [12], and so the universal $C^*$-algebra $C^*(\Gamma)$ coincides with the $C^*$-algebra of $\Gamma$ regarded as a $k$-graph. To see that the generators of $C^*(\Gamma)$ are all nonzero, we introduce the groupoid $\mathcal{G}_\Gamma$.

For each $\xi \in \Gamma$, define $Z(\xi) \subseteq \Gamma^\Omega$ by

$$Z(\xi) := \{ x \in \Gamma^\Omega : x(0, d(\xi)) = \xi \}.$$

The $Z(\xi)$ constitute a base of compact open sets for a second-countable locally compact Hausdorff topology on $\Gamma^\Omega$. For $x \in \Gamma^\Omega$ and $p \in P$, there is a unique element $\sigma^p(x)$ of $\Gamma^\Omega$ defined by $\sigma^p(x)(g, r) = x(p + q, p + r)$. Define $\mathcal{G}_\Gamma := \{ (x, p - q, y) : x, y \in \Omega^P, \sigma^p(x) = \sigma^q(y) \}$. This is a groupoid with $r(x, g, y) = (x, 0, x)$, $s(x, g, y) = (y, 0, y)$, $(x, g, y)(y, h, z) = (x, g + h, z)$ and $(x, g, y)^{-1} = (y, -g, x)$. Arguments like those of [12]...
show that \( \mathcal{G}_r \) is a locally compact Hausdorff étale groupoid under the topology generated by the sets

\[
Z(\xi, \eta) = \{(x, d(\xi) - d(\eta), y) : x \in Z(\xi), y \in Z(\eta), \sigma^d(\xi)(x) = \sigma^d(\eta)(y)\}.
\]

Each \( Z(\xi, \eta) \) is compact open in \( \mathcal{G}_r \).

We recall from [20] the construction of \( C_r^*(\mathcal{G}_r) \). The space \( C_c(\mathcal{G}_r) \) is a *-algebra under the convolution product \((a * b)(x, y) = \sum (x, h, z) \in \mathcal{G}_r \ a(x, h, z) b(z, h^{-1}, y)\) and the involution \( a^*(x, y) = a(y, g^{-1}, x)\). For \( y \in \Gamma^0 \), let \( \mathcal{H}_y = l^2(\{s(\alpha) = y\}) \) with orthonormal basis \( \{\delta_s : s \in \mathcal{G}_r, s(\alpha) = y\} \). The regular representation \( \rho_y \) of \( C_c(\mathcal{G}_r) \) on \( \mathcal{H}_y \) is given by \( \rho_y(a) \delta_{(x, g, y)} = \sum_{(z, h, x) \in \mathcal{G}_r} a(z, h, x) \delta_{(z, h, g, y)} \). The reduced groupoid \( C_r^*(\mathcal{G}_r) \) is the closure of the image of \( C_c(\Gamma) \) under \( \bigoplus_y \rho_y \); equivalently, it is the completion of \( C_c(\mathcal{G}_r) \) in the norm \( \|a\| = \sup_y \|ho_y(a)\| \).

**Lemma 2.5.** Let \( P \) be a finitely generated cancellative abelian monoid. Let \( \Gamma \) be a row-finite \( P \)-graph with no sources. For \( \xi \in \Gamma \), let \( t_\xi := 1_{Z(\xi, s(\xi))} \in C_c(\mathcal{G}_r) \subseteq C_r^*(\mathcal{G}_r) \). Then \( \{t_\xi : \xi \in \Gamma\} \) is a Cuntz-Krieger \( \Gamma \)-family, and there is a homomorphism \( \pi_\Gamma : C^*(\Gamma) \to C_r^*(\mathcal{G}_r) \) which carries each \( s_\xi \) to \( 1_{Z(\xi, s(\xi))} \). In particular, each generator of \( C^*(\Gamma) \) is nonzero.

**Proof.** Routine calculations show that the \( t_\xi \) form a Cuntz-Krieger \( \Gamma \)-family. The existence of the homomorphism \( \pi_\Gamma \) then follows from the universal property of \( C^*(\Gamma) \). Since the \( Z(\xi, s(\xi)) \) are all nonempty, the \( t_\xi \) are all nonzero and hence the \( s_\xi \) are also nonzero. \( \square \)

If \( G \) is the Grothendieck group of \( P \), then for each character \( \chi \) of \( G \), the elements \( \gamma_\chi(s_\xi) := \chi(d(\xi)) s_\xi \in C^*(\Gamma) \) form a Cuntz-Krieger \( \Gamma \)-family, and hence induce an endomorphism \( \gamma_\chi \) of \( C^*(\Gamma) \). Since \( \gamma_{\chi^{-1}} \) is an inverse for \( \gamma_\chi \), both are automorphisms. An \( \varepsilon/3 \)-argument shows that \( \gamma \) is a continuous action of \( G \) by automorphisms. Averaging over \( \gamma \) gives a faithful conditional expectation \( \Phi \) onto the fixed-point algebra \( C_r^*(\Gamma)^\gamma \) [10, Proposition 3.2]. Since every spanning element of \( C^*(\Gamma) \) belongs to one of the spectral subspaces of \( \gamma \), we have

\[
\Phi(s_\xi s_\eta^*) = \delta_{d(\xi), d(\eta)} s_\xi s_\eta^*.
\]

In particular, we have \( C^*(\Gamma)^\gamma = \overline{\text{span}\{s_\xi s_\eta^* : d(\xi) = d(\eta)\}} \).

For a countable set \( X \), we write \( K_X \) for the unique nonzero \( C^* \)-algebra generated by elements \( \{\theta_{x,y} : x, y \in X\} \) such that \( \theta_{x,y} = \theta_{y,x} \) and \( \theta_{x,y} \theta_{w,z} = \delta_{y,w} \theta_{x,z} \).

**Lemma 2.6.** Let \( P \) be a finitely generated cancellative abelian monoid and let \( \Gamma \) be a row-finite \( P \)-graph with no sources. Then \( C^*(\Gamma)^\gamma \) is an AF algebra. A homomorphism \( \phi : C^*(\Gamma) \to B \) restricts to an injection of \( C^*(\Gamma)^\gamma \) if and only if \( \phi(s_v) \neq 0 \) for all \( v \).

**Proof.** Let \( a_1, \ldots, a_k \) be generators for \( P \). For each \( n \in \mathbb{N} \), let \( n \cdot 1 := \sum_{i=1}^k n_i a_i \), and let \( A_n := \text{span}\{s_\xi s_\eta^* : d(\xi) = n \cdot 1\} \). The Cuntz-Krieger relations ensure that the \( s_\xi s_\eta^* \) are matrix units. By Lemma 2.5, all the \( s_\xi \) are nonzero, and hence \( s(\xi) = s(\eta) \) implies \( \|s_\xi s_\eta^*\|^2 = \|s_\eta s_\xi^* s_\xi s_\eta^*\| = \|s_\eta^* s_\xi s_\eta\| = \|s_\eta^* s_\eta\| \neq 0 \). It follows that \( s_\xi s_\eta^* \mapsto \Theta_{\xi, \eta} \) determines an isomorphism \( A_n \cong \bigoplus_{v \in \Gamma^0} K_{\Gamma^0}^{1,v} \). If \( m \leq n \), then for \( v \in \Gamma^0 \) and \( \xi, \eta \in \Gamma^0 \), we have

\[
s_\xi s_\eta^* = \sum_{\zeta \in \Gamma^0(n-m) \cdot 1} s_\xi s_\eta^* \in A_n,
\]

and hence \( m \leq n \) implies \( A_m \subseteq A_n \). Hence \( \bigcup_n A_n \) is an AF subalgebra of \( C^*(\Gamma)^\gamma \). Suppose that \( \xi, \eta \in \Gamma \) satisfy \( s(\xi) = s(\eta) = v \) and \( d(\xi) = d(\eta) \). Write \( d(\xi) = \sum_{i=1}^k p_i a_i \). Let \( n := \)
Lemma 2.6 implies that \( C_t \) each
\[ \text{\( \in \bigcup_n A_n \) is AF.} \]

Now suppose that \( \phi : C^*(\Gamma) \to B \) is a homomorphism. Since each \( s_v \) is nonzero and belongs to \( C^*(\Gamma)^\gamma \), if \( \phi \) is injective on \( C^*(\Gamma)^\gamma \), then each \( \phi(s_v) \neq 0 \). Conversely suppose that each \( \phi(s_v) \neq 0 \). Fix \( n \in \mathbb{N} \). For \( \xi, \eta \in \Gamma^{n-1} \), the reasoning of the first paragraph of this proof shows that \( \phi(s_v s_v^*) \neq 0 \). Since each \( K_{\Gamma-n-v} \) is simple, it follows that \( \phi \) is injective and hence isometric on \( A_n \). Since the \( A_n \) are nested, it follows that \( \phi \) is isometric on \( \bigcup_n A_n \), and therefore on \( \bigcup_n A_n = C^*(\Gamma)^\gamma \) as well. \( \square \)

**Proposition 2.7.** Let \( P \) be a finitely generated cancellative abelian monoid, and let \( G \) be its Grothendieck group. Let \( \Gamma \) be a row-finite \( P \)-graph with no sources. Suppose that \( \{t_\xi : \xi \in \Gamma\} \) is a Cuntz-Krieger \( \Gamma \)-family in a \( C^* \)-algebra \( B \) and there is an action \( \beta \) of \( \hat{\Gamma} \) on \( B \) such that \( \beta_\chi(t_\xi) = \chi(d(\xi))t_\xi \) for all \( \xi \in \Gamma \). Then the induced homomorphism \( \pi_\chi : C^*(\Gamma) \to B \) is injective if and only if \( t_\chi \neq 0 \) for all \( \chi \in \Gamma^0 \). The homomorphism \( \pi_\chi \) of Lemma 2.5 is an isomorphism from \( C^*(\Gamma) \) to \( C^*_\chi(G_\Gamma) \).

**Proof.** First observe that if some \( t_\chi = 0 \) then Lemma 2.5 implies that \( \pi_\chi \) is not injective. Now suppose that each \( t_\chi \neq 0 \). By Lemma 2.6, the homomorphism \( \pi_\chi \) is injective on \( C^*(\Gamma)^\gamma \). Averaging over \( \beta \) gives a conditional expectation \( \Psi : \pi_\chi(C^*(\Gamma)) \to \pi_\chi(C^*(\Gamma)^\gamma) \) such that \( \Psi \circ \pi_\chi = \pi_\chi \circ \Phi \), where \( \Phi \) is the conditional expectation of \( \mathbb{I} \). Now the following standard argument shows that \( \pi_\chi \) is faithful:

\[
\pi_\chi(a) = 0 \implies \Psi(\pi_\chi(a^*a)) = 0 \implies \pi_\chi(\Phi(a^*a)) = 0 \implies \Phi(a^*a) = 0 \implies a = 0.
\]

Now let \( \{t_\xi : \xi \in \Gamma\} \) and \( \pi_\chi : C^*(\Gamma) \to C^*_\chi(G_\Gamma) \) be as in Lemma 2.5. Define \( c : G_\Gamma \to G \) by \( c(x, y, g) = g \). For \( n \in \mathbb{N} \), let \( \rho_y \) be the regular representation discussed prior to Lemma 2.5. For \( \alpha, \beta \in G_\Gamma \) with \( s(\alpha) = s(\beta) = y \), let \( \Theta_{\alpha, \beta} \in B(H_y) \) be the rank-one operator from \( \mathbb{C}d_\beta \) to \( \mathbb{C}d_\alpha \). There is a strongly continuous action \( \beta^y \) of \( \hat{\Gamma} \) on \( B(H_y) \) such that \( \beta_\chi^y(\Theta_{\alpha, \beta}) = \chi(c(\alpha) - c(\beta))\Theta_{\alpha, \beta} \). In particular, \( \beta_\chi^y(\rho_y(t_\xi)) = \chi(d(\xi))\rho_y(t_\xi) \) for all \( \chi \in \Gamma \) and \( \xi \in \hat{\Gamma} \). Thus \( \beta = \bigoplus_y \beta^y \) is a strongly continuous action of \( \hat{\Gamma} \) on \( C^*_\chi(G_\Gamma) \) such that \( \beta_\chi(t_\xi) = \chi(d(\xi))t_\xi \) for all \( \chi \in \Gamma \). So \( \pi_\chi \) is injective. Lemma 2.5 implies that if \( s(\xi) = s(\eta) \) in \( \Gamma \), then \( \pi_\chi(s_\xi s_\eta^*) = \{1 \in \xi \} \{1 \in \eta \} = 1 \{1 \in \xi \} \{1 \in \eta \} \). The \( C^* \)-norm on \( C^*_\chi(G_\Gamma) \) coincides with the supremum norm on each \( C_c(Z(\xi, \eta)) \), so the Stone-Weierstrass theorem implies that the range of \( \pi_\chi \) contains \( C_c(G_\Gamma) \), and hence \( \pi_\chi \) is surjective. \( \square \)

Let \( P \) be a finitely generated cancellative abelian monoid and let \( \Gamma \) be a \( P \)-graph. We say that \( \Gamma \) is **aperiodic** if for every \( \chi \in \Gamma^0 \) there exists \( x \in \Gamma^\omega \) such that \( p \neq q \in P \) implies \( \sigma^p(x) \neq \sigma^q(x) \).

**Corollary 2.8.** Let \( P \) be a finitely generated abelian monoid and let \( \Gamma \) be a row-finite \( P \)-graph with no sources. Suppose that \( \Gamma \) is aperiodic. If \( \{t_\xi : \xi \in \Gamma\} \) is a Cuntz-Krieger \( \Gamma \)-family in a \( C^* \)-algebra \( B \), then the induced homomorphism \( \pi_\chi : C^*(\Gamma) \to B \) is injective if and only if \( t_\chi \neq 0 \) for every \( \chi \in \Gamma^0 \).

**Proof.** The only if is clear because Lemma 2.5 implies that each \( s_v \neq 0 \). Suppose that each \( t_\chi \neq 0 \). Then each \( t_\chi t_\chi^\star \neq 0 \). By Proposition 2.7, we can identify \( C^*(\Gamma) \) with \( C^*_\chi(G_\Gamma) \), and regard \( \pi_\chi \) as a homomorphism of \( C^*_\chi(G_\Gamma) \) which carries each \( 1 \{1 \in \xi \} \) to \( t_\xi \). The isomorphism \( C^*(\Gamma) \cong C^*_\chi(G_\Gamma) \) identifies \( C_0(G_\Gamma(0)) \) with a subalgebra of \( C^*(\Gamma)^\gamma \), so Lemma 2.6 implies that \( \pi_\chi \) is injective on \( C_0(G_\Gamma(0)) \).

Fix a basic open set \( Z(\xi) \) in \( G_\Gamma(0) = \Gamma^\omega \). By hypothesis, there exists \( y \in \Gamma^\omega \) such that \( y(0) = s(\xi) \) and \( \sigma^p(y) \neq \sigma^q(y) \) whenever \( p \neq q \). The factorisation property implies that
there is a unique element $\xi y$ of $Z(\xi)$ such that $\sigma^d(\xi y) = y$. We have
\[\sigma^p(\xi y) = \sigma^q(\xi y) \implies \sigma^d(\xi y) = \sigma^d(\xi y) \implies \sigma^p(y) = \sigma^q(y) \implies p = q.\]
Thus $G_\Lambda$ has trivial isotropy at $\xi y$. Thus $G_T$ is topologically principal. It now follows from \cite{7} Theorem 4.4 that every nontrivial ideal of $C^*_v(G_T)$ has nontrivial intersection with $C_0(G_T^{(0)})$. In particular, that $\pi_t|_{C_0(G_T^{(0)})}$ is injective shows that $\pi_t$ is injective. \hfill $\square$

3. THE PRIMITIVE IDEAL SPACE OF THE $C^*$-ALGEBRA OF A PULLBACK

In this section we consider pullback $k$-graphs of the form $f^*\Gamma$ where $f : \mathbb{Z}^k \to G$ is a group homomorphism and $\Gamma$ is a $P$-graph for $P = f(\mathbb{N}^k)$. Proposition 3.3 and Lemma 3.4 combine to show that, putting $H := \ker(f)$, the $C^*$-algebra $C^*(f^*\Gamma)$ is a $C(\hat{H})$-algebra with fibres identical to $C^*(\Gamma)$. We use this to give a complete listing of the irreducible representations of $C^*(f^*\Gamma)$ in terms of the irreducible representations of $C^*(\Gamma)$ and characters of $\ker f$. We begin by introducing pullbacks of $P$-graphs.

Definition 3.1 (cf. \cite{12} Definition 1.9). Let $P$ and $Q$ be finitely generated cancellative abelian monoids, and let $f : P \to Q$ be a monoid morphism. If $(\Gamma, d)$ is a $Q$-graph, we define the $P$-graph $f^*\Gamma$ as follows: $f^*\Gamma = \{ (\lambda, n) : d(\lambda) = f(n) \}$ with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$ and $r(\lambda, n) = r(\lambda)$. Composition is given by $(\mu, m)(\nu, n) = (\mu \nu, m + n)$.

For $g, h \in \mathbb{Z}^k$, we write $g \lor h$ for the coordinatewise maximum of $g$ and $h$ and $g \land h$ for the coordinatewise minimum. Given $h \in \mathbb{Z}^k$ we define $h_+ := h \lor 0$ and $h_- := -(h \land 0)$. We then have $h = h_+ - h_-$ with $h_+ \lor h_- = 0$.

Lemma 3.2. Let $H$ be a subgroup of $\mathbb{Z}^k$, let $G = \mathbb{Z}^k/H$ and let $f : \mathbb{Z}^k \to G$ be the quotient map. Let $P = f(\mathbb{N}^k) \subseteq G$. Suppose that $\Gamma$ is a row-finite $P$-graph with no sources. Then $f^*\Gamma$ is a row-finite $k$-graph with no sources. Suppose that $h \in H$ and $\lambda \in \Gamma$ satisfy $d(\lambda) = f(h_+)$. Then $(\lambda, h_+)$ and $(\lambda, h_-)$ both belong to $f^*\Gamma$.

Proof. Clearly $f^*\Gamma$ is a countable category and $(\lambda, n) \mapsto n$ is a functor. We check the factorisation property: if $(\lambda, m + n) \in f^*\Gamma$, then $d(\lambda) = f(m) + f(n)$. So $\lambda$ factorises uniquely as $\lambda = \mu \nu$ with $d(\mu) = f(m), d(\nu) = f(n)$, and then $(\lambda, m + n) = (\mu, m)(\nu, n)$ is the unique factorisation with $d(\mu, m) = m$ and $d(\nu, n) = n$.

For the second assertion, observe that $f(h_+) = f(h_- + h) = f(h_-) + f(h) = f(h_-)$ since $h \in H = \ker(f)$. \hfill $\square$

In the following proposition, $\{ U_h : h \in H \}$ denotes the canonical collection of unitary generators of the group $C^*$-algebra $C^*(H)$. We write $ZM(A)$ for the centre of the multiplier algebra of a $C^*$-algebra $A$.

Proposition 3.3. Let $G, f, P, H$ be as in Lemma 3.3. Let $\Gamma$ be a row-finite $P$-graph with no sources. For each $h \in H$ and $v \in \Gamma^0$, let $u_{v, h} := \sum_{\lambda \in \Gamma^0} s(\lambda, h_+^+) s(\lambda, h_-^+)$. Then $\sum_{v \in \Gamma^0} u_{v, h}$ converges strictly to a central unitary multiplier $V_h := \sum_{\lambda} s(\lambda, h_+) s(\lambda, h_-^+)$ of $C^*(f^*\Gamma)$. Moreover, there is an injective homomorphism $\rho : C^*(H) \to ZM(C^*(f^*\Gamma))$ such that $\rho(U_h) = V_h$ for all $h \in H$.

Proof. For $v \in \Gamma^0$ and $h \in H$, we claim that $u_{v, h}$ is a partial isometry whose initial and final projections are both equal to $s(v, 0)$. Indeed
\[
    u_{v, h} u_{v, h}^* = \sum_{(\lambda, \mu) \in \Gamma^0} s(\lambda, h_-^+) s(\lambda, h_+^+) s(\mu, h_+^+) s(\mu, h_-^+) = \sum_{\eta \in (v, 0)} s(\eta) s(\eta)^* = s(v, 0).
\]
and a similar calculation shows that \( u_{v,h}^* u_{v,h} = s_{(v,0)} \) also.

By, for example, the argument of [16, Lemma 2.10], for each \( a \in C^*(f^*\Gamma) \), we have \( \sum_{v \in F} s_{(v,0)} a \to a \) as \( F \) increases over finite subsets of \( \Gamma^0 \). So for \( a \in C^*(f^*\Gamma) \) and \( \varepsilon > 0 \), there exists a finite \( F \subseteq \Gamma^0 \) such that \( \|a - \sum_{v \in F} s_v a\| < \varepsilon \). Thus, for \( K \subseteq L \subseteq \Gamma^0 \setminus F \) and \( h \in H \), we have

\[
\left\| \sum_{v \in L} u_{v,h} a - \sum_{w \in K} u_{w,h} a \right\| = \left\| \sum_{v \in L \setminus K} u_{v,h} a \right\| \\
= \left\| \sum_{v \in L \setminus K} u_{v,h} \left( a - \sum_{w \in F} s_{(w,0)} a \right) \right\| \text{ since } L \subseteq \Gamma^0 \setminus F \\
\leq \varepsilon \left\| \sum_{v \in L \setminus K} u_{v,h} \right\| \\
= \varepsilon
\]

since the \( u_{v,h} \) are partial isometries whose initial projections are mutually orthogonal and whose final projections are also mutually orthogonal. Hence the net \( \left\{ \sum_{v \in F} u_{v,h} a : F \subseteq \Gamma^0 \text{ is finite} \right\} \) is Cauchy, so converges. Thus, the series

\[
\sum_{v \in \Lambda^0} u_{v,h} = \sum_{d(\lambda) = f(h_+)} \lambda \cdot \sigma(\lambda,h_+) \cdot \sigma(\lambda,h_-)
\]

converges strictly to a multiplier \( V_h \) of \( C^*(f^*\Gamma) \).

Since \( (h^-)_+ = h_- \) and \( (h^-)_- = h_+ \) for all \( h \in \mathbb{Z}^k \), we have \( u_{v,h}^* = u_{v,-h} \) for all \( v, h \). Hence \( V_h^* = V_{-h} \) for all \( h \in H \). The \( V_h \) are unitary because

\[
(V_h V_h^* a) = \lim_{F \uparrow K} \sum_{v \in F, w \in K} u_{v,h} u_{w,h} a = \lim_{F \uparrow K} \sum_{v \in F} s_{(v,0)} a = a,
\]

so \( V_h^* V_h = 1_{\mathcal{M}(C^*(f^*\Gamma))} \), and then \( V_h^* V_h = V_{-h} V_{-h} = 1_{\mathcal{M}(C^*(f^*\Gamma))} \) also.

To see that the \( V_h \) are central, observe that

\[
V_h s_{(\lambda,m)} = \lim_{F \uparrow K} \sum_{v \in F} u_{v,h} s_{(\lambda,m)} = \lim_{F \uparrow K} \sum_{v \in F} \sum_{\mu \in \Gamma^f(h_+)} s_{(\mu,h_+)} s_{(\lambda,m)} s_{(\mu,h_-)} s_{(\lambda,m)}.
\]

Applying the Cuntz-Krieger relation, we obtain

\[
V_h s_{(\lambda,m)} = \lim_{F \uparrow K} \sum_{\mu \in \Gamma^f(h_+)} \sum_{\alpha \in s(\mu) \Gamma^f(h_-)} s_{(\mu,\alpha,h_+ + m)} s_{(\mu,\lambda,h_- + m)} s_{(\lambda,m)}.
\]

The only nonzero terms are those where \( \mu \alpha = \lambda \beta \). Since \( \Gamma \) has no sources, for each \( \beta \in s(\lambda) \Gamma^f(h_-) \) there is a unique \( \mu \in r(\lambda) \Gamma^f(h_+ - m) \) and a unique \( \alpha \in s(\mu) \Gamma^f(h_-) \) such that \( \mu \alpha = \lambda \beta \). So the final sum above collapses to give

\[
V_h s_{(\lambda,m)} = \sum_{\beta \in s(\lambda) \Gamma^f(h_-)} s_{(\lambda,\beta,\lambda,h_+ + m)} s_{(\beta,\lambda,h_-)}.
\]

On the other hand,

\[
s_{(\lambda,m)} V_h = s_{(\lambda,m)} \lim_{F \uparrow K} \sum_{\beta \in \Gamma^f(h_+)} s_{(\lambda,\beta,\lambda,h_+ + m)} s_{(\beta,\lambda,h_-)}
\]

\[
= \sum_{\beta \in s(\lambda) \Gamma^f(h_+)} s_{(\lambda,m)} s_{(\lambda,\beta,\lambda,h_+ + m)} s_{(\beta,\lambda,h_-)} = \sum_{\beta \in s(\lambda) \Gamma^f(h_+)} s_{(\lambda,\beta,\lambda,h_+ + m)} s_{(\beta,\lambda,h_-)}
\]

as required.
The universal property of $C^*(H)$ implies that there is a homomorphism $\rho : C^*(H) \to \mathcal{ZM}(C^*(f^*\Gamma))$ such that $\rho(U_h) = V_h$ for all $h \in H$. To see that $\rho$ is injective, define an action $\beta$ of $\hat{H}$ on $C^*(f^*\Gamma)$ by $\beta_\lambda(s_{(\lambda,n)}) = \chi(n)s_{(\lambda,n)}$. Then $\beta$ extends to an action of $\hat{H}$ on $\mathcal{ZM}(C^*(f^*\Gamma))$ such that $\beta(V_h) = \chi(h) V_h$. So $\rho$ is nonzero and equivariant for $\beta$ and the dual action of $\hat{H}$ on $C^*(H)$. Therefore $\rho$ is injective.

**Lemma 3.4.** Let $G, f, P, H$ be as in Lemma 3.3. Let $\Gamma$ be a row-finite P-graph with no sources. For $z \in \mathbb{T}^k$, let $I_z$ be the ideal of $C^*(f^*\Gamma)$ generated by $\{z^{-n_1}s_{(\lambda_1,n_1)} - z^{-n_2}s_{(\lambda_2,n_2)} : \lambda \in \Gamma, n_1, n_2 \in \mathbb{N}^k, f(n_1) = f(n_2) = d(\lambda)\}$. Then there is an isomorphism $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ such that $\psi_z(s_{(\lambda,n)} + I_z) = z^n s_\lambda$ for all $\lambda \in \Gamma$.

**Proof.**
1. The set $\{z^n s_\lambda : (\lambda, n) \in f^*\Gamma\}$ is a Cuntz-Krieger $f^*\Gamma$-family. Hence there is a homomorphism $\psi : C^*(f^*\Gamma) \to C^*(\Gamma)$ carrying each $s_{(\lambda,n)}$ to $z^n s_\lambda$. Each generator of $I_z$ belongs to $\ker \psi$, and hence $\psi$ descends to a homomorphism $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ satisfying $\psi_z(s_{(\lambda,n)} + I_z) = z^n s_\lambda$.

To see that this $\psi_z$ is an isomorphism, we show that it has an inverse. Fix $\lambda \in \Gamma$ and $n_1, n_2 \in \mathbb{N}^k$ such that $f(n_1) = f(n_2) = d(\lambda)$. Then $z^{-n_1}s_{(\lambda_1,n_1)} - z^{-n_2}s_{(\lambda_2,n_2)} \in I_z$. So we may define a collection $\{t_\lambda : \lambda \in \Gamma\} \subseteq C^*(f^*\Gamma)/I_z$ by $t_\lambda = z^{-n}s_{(\lambda,n)} + I_z$ for any $n \in f^{-1}(d(\lambda))$; in particular $t_\lambda$ is nonzero and equivariant for $\lambda$.

Moreover, $\psi_z(s_{(\lambda,n)} + I_z) = z^n s_\lambda$ for any $n \in f^{-1}(d(\lambda))$; in particular $t_\lambda$ is an irreducible representation of $\hat{H}$. To see that this $\psi_z$ is an inverse for $\psi_z$.

In the following theorem, given a $C^*$-algebra $A$, we write $\text{Irr}(A)$ for the collection of all irreducible representations of $A$.

**Theorem 3.5.** Let $G$ be a finitely generated abelian group, and let $f : \mathbb{Z}^k \to G$ be a homomorphism. Let $P = f(\mathbb{N}^k) \subseteq G$ and let $H = \ker(f) \subseteq \mathbb{Z}^k$. Let $\Gamma$ be a row-finite P-graph with no sources. For $z \in \mathbb{T}^k$, let $q_z : C^*(f^*\Gamma) \to C^*(f^*\Gamma)/I_z$ be the quotient map, and let $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ be the isomorphism of Lemma 3.4. Let $\pi$ be an irreducible representation of $C^*(\Gamma)$. Then $\pi \circ \psi_z \circ q_z$ is an irreducible representation of $C^*(f^*\Gamma)$. Fix a map $\gamma : \mathbb{T}^k \to \mathbb{T}^k$ such that $z^n = \gamma(h)$ for all $\gamma \in \hat{H}$ and $h \in H$. Then $(\gamma, \pi) \mapsto \pi \circ \psi_{z^n} \circ q_{z^n}$ is a bijection of $\hat{H} \times \text{Irr}(C^*(\Gamma))$ onto $\text{Irr}(C^*(f^*\Gamma))$.

**Remark 3.6.** Theorem 3.5 together with the definition of $I_z$ implies that if $\pi$ is an irreducible representation of $C^*(f^*\Gamma)$, then there is a unique character $\gamma$ of $\hat{H}$ such that $s_{(\lambda_1,n_1)} - s_{(\lambda_2,n_2)} \in \ker(\pi)$ for all $\lambda \in \Gamma$ and $n_1, n_2 \in f^{-1}(d(\lambda))$.

We collect some more technical lemmas before proving Theorem 3.5.

**Lemma 3.7.** Let $G, f, P, H$ be as in Lemma 3.3. Let $\Gamma$ be a row-finite P-graph with no sources. For $z \in \mathbb{T}^k$, let $I_z$ be the ideal of $C^*(f^*\Gamma)$ generated by $\{z^{-n_1}s_{(\lambda_1,n_1)} - z^{-n_2}s_{(\lambda_2,n_2)} : \lambda \in \Gamma, n_1, n_2 \in f^{-1}(d(\lambda))\}$.
\( \lambda \in \Gamma, \ n_1, n_2 \in \mathbb{N}^k, \ f(n_1) = f(n_2) = d(\lambda) \}. \) For \( w, z \in \mathbb{T}^k \) we have \( I_w = I_z \) if and only if \( w^h = z^h \) for all \( h \in H \).

**Proof.** If \( w^h = z^h \) for all \( h \in H \), then \( I_w = I_z \) because they have the same generators. Now suppose that \( I_w = I_z \). Fix \( h \in H \). For \( v \in \Gamma^0 \) we have \( w^h s(v,0) - u_{v,h} \in I_w = I_z \) for all \( h \in H \). Subtracting, we obtain \( (w^h - z^h) s(v,0) \in I_z \). The isomorphism \( \psi_z \) of Lemma 3.3 carries each \( s(v,0) + I_z \) to \( s(w,0) \) which is nonzero by Proposition 2.7. Hence \( s(v,0) \not\in I_z \), which forces \( w^h - z^h = 0 \). Hence \( w^h = z^h \) for all \( h \in H \).

**Lemma 3.8.** Let \( G, f, P, H \) be as in Lemma 3.3. Let \( \Gamma \) be a row-finite \( P \)-graph with no sources. Let \( \varphi \) be an irreducible representation of \( C^*(f \Gamma) \) on a Hilbert space \( \mathcal{H} \). Then \( \varphi \) has a unique strict extension to a representation \( \tilde{\varphi} \) of \( MC^*(f^* \Gamma) \) and there is a unique character \( \gamma_\varphi \) of \( H \) such that \( \tilde{\varphi}(V_h) = \gamma_\varphi(h)1_\mathcal{H} \) for all \( h \in H \). For any \( z \in \mathbb{T}^k \) such that \( z^h = \gamma_\varphi(h) \) for all \( h \in H \), we have \( I_z \subseteq \ker(\varphi) \). There is an irreducible representation \( \tilde{\varphi} \) of \( C^*(f^* \Gamma) / I_z \) such that \( \tilde{\varphi}(a + I_z) = \gamma_\varphi(a) \) for all \( a \), and \( \tilde{\varphi} \circ \psi_z^{-1} \) is an irreducible representation of \( C^*(\Gamma) \).

**Proof.** The irreducible representation \( \varphi : C^*(f^* \Gamma) \to B(\mathcal{H}) \) is extendible since it is non-degenerate. Since \( \varphi \) is irreducible, \( \tilde{\varphi} \) is also. Since the \( V_h \) are central in \( MC^*(f^* \Gamma) \), the \( \tilde{\varphi}(V_h) \) are central in \( \tilde{\varphi}(MC^*(f^* \Gamma)) \). Therefore the \( \tilde{\varphi}(V_h) \) are all scalar multiples of identity operator \( 1_\mathcal{H} \) by, for example, [13] Theorem 4.1.12. That is, for each \( h \in H \), there exists \( \gamma_\varphi(h) \in \mathbb{T} \) such that \( \tilde{\varphi}(V_h) = \gamma_\varphi(h)1_\mathcal{H} \). Since \( \tilde{\varphi} \) is multiplicative, \( \gamma_\varphi : H \to \mathbb{T} \) is a homomorphism, so \( \gamma_\varphi \in \hat{H} \). The uniqueness of \( \gamma_\varphi \) is obvious.

Fix \( z \in \mathbb{T}^k \) such that \( z^h = \gamma_\varphi(h) \) for all \( h \in H \). For \( h \in H \), we have

\[
\varphi((z^h - u_{v,h}) s(v,0)) = \tilde{\varphi}(z^h s(v,0) - u_{v,h}) = 0.
\]

So \( \varphi \) descends to a representation \( \tilde{\varphi} \) of \( C^*(f^* \Gamma) / I_z \), which is irreducible because \( \varphi \) is.

**Proof of Theorem 3.3** Since the homomorphism \( \psi_z : C^*(f^* \Gamma) / I_z \to C^*(\Gamma) \) of Lemma 3.3 is an isomorphism, the composition \( \psi_z \circ q_z \) is surjective, and hence \( \pi \circ \psi_z \circ q_z \) is an irreducible representation of \( C^*(f^* \Gamma) \).

To see that \( (\gamma, \pi) \mapsto \pi \circ \psi_z \circ q_{z^\gamma} \) is surjective, fix an irreducible representation \( \varphi \) of \( C^*(f^* \Gamma) \). Let \( \gamma_\varphi \) be the corresponding character of \( H \) given in Lemma 3.3 and let \( \psi_z^{\gamma_\varphi} : C^*(f^* \Gamma) / I_z \to C^*(\Gamma) \) be the isomorphism of Lemma 3.3. By Lemma 3.3, there is an irreducible representation \( \tilde{\varphi} \) of \( C^*(f^* \Gamma) / I_z \) such that \( \tilde{\varphi} \circ q_{z^\gamma} = \varphi \), and then \( \tilde{\varphi} \circ \psi_z^{\gamma_\varphi}^{-1} \) is an irreducible representation of \( C^*(\Gamma) \). Now

\[
(\tilde{\varphi} \circ \psi_z^{\gamma_\varphi}^{-1}) \circ \psi_z^{\gamma_\varphi} \circ q_{z^\gamma} = \tilde{\varphi} \circ q_{z^\gamma} = \varphi,
\]

so \( (\gamma, \pi) \mapsto \pi \circ \psi_z \circ q_z \) is surjective.

To see that it is injective, fix \( \pi \) and \( \gamma \). We claim that for \( z \in \mathbb{T}^k \), we have \( I_z \subseteq \ker(\pi \circ \psi_z \circ q_z) \) if and only if \( z^h = \gamma(h) \) for all \( h \in H \). First suppose that \( I_z \subseteq \ker(\pi \circ \psi_z \circ q_z) \). Fix \( h \in H \), and let \( \theta = \pi \circ \psi_z \circ q_z \). Then \( \theta(u_{v,h}) = z^h \pi(s(v)) \). Hence

\[
0 = \theta(z^h s(v,0) - u_{v,h}) = (z^h - \gamma(h)) \pi(s(v)).
\]

Since \( \pi \) is nonzero, there exists \( v \in \Gamma^0 \) such that \( \pi(s(v)) \neq 0 \) forcing \( z^h = \gamma(h) \) for all \( h \in H \). Now suppose that \( z^h = \gamma(h) \) for all \( h \in H \). Then every generator of \( I_z \) belongs to \( \ker \theta \), giving \( I_z \subseteq \ker \theta \).

Fix \( (\gamma_1, \pi_1), (\gamma_2, \pi_2) \in \hat{H} \times \text{Irr}(C^*(\Gamma)) \), and write \( \theta_i := \pi_i \circ \psi_{z^1_i} \circ q_{z^1_i} \) for \( i = 1, 2 \). Suppose that \( \theta_1 = \theta_2 \). Then \( I_z \subseteq \ker(\theta_1) = \ker(\theta_2) \) for all \( z \in \mathbb{T}^k \). Thus \( \gamma_1(h) = \gamma_2(h) \) for all \( h \in H \) by the preceding paragraph; that is \( \gamma_1 = \gamma_2 \). This forces \( \psi_{z^1_i} \circ q_{z^1_i} = \psi_{z^2_i} \circ q_{z^2_i} \).

Since these are surjections onto \( C^*(\Gamma) \), the equality \( \theta_1 = \theta_2 \) forces \( \pi_1 = \pi_2 \).
4. Maximal tails, periodicity, and pullbacks

Theorem 3.12 of [11] implies that the maximal tails in a strongly aperiodic $k$-graph index the primitive gauge-invariant ideals in its $C^*$-algebra. These maximal tails are also a key ingredient in our catalogue of the primitive ideals of an arbitrary $k$-graph $C^*$-algebra. In this section we show that every maximal tail $T$ (Definition 3.10) in turn is the key to our main theorem in Section 5. This implies that for gauge-invariant ideals associated to maximal tails, the quotient contains a corner whose primitive-ideal space is described by Theorem 3.5. This in turn is the key to our main theorem in Section 5.

We recall the definition of a maximal tail from [11].

**Definition 4.1** ([11 Definition 3.10]). Let $\Lambda$ be a row-finite $k$-graph with no sources. A nonempty subset $T$ of $\Lambda^0$ is called a maximal tail if

(a) for every $v_1, v_2 \in T$ there is $w \in T$ such that $v_1 \Lambda w \neq \emptyset$ and $v_2 \Lambda w \neq \emptyset$,

(b) for every $v \in T$ and $1 \leq i \leq k$ there exists $\lambda \in v\Lambda^0$ such that $s(\lambda) \in T$, and

(c) for $w \in T$ and $v \in \Lambda^0$ with $v\Lambda w \neq \emptyset$ we have $v \in T$.

In [5], Davidson and Yang comprehensively analyse aperiodicity for single-vertex 2-graphs. The following results substantially generalise this analysis, but the fundamental idea behind them is due to Davidson-Yang.

Let $\Lambda$ be a row-finite $k$-graph with no sources such that $\Lambda^0$ is a maximal tail. We define a relation on $\Lambda$ by

$$\mu \sim \nu \quad \text{if and only if} \quad s(\mu) = s(\nu) \quad \text{and} \quad \mu x = \nu x \quad \text{for all} \quad x \in s(\mu)\Lambda^\infty.$$  

This is an equivalence relation on $\Lambda$ which respects range, source and composition. Thus $\Lambda/\sim$ is a category with respect to $r([\lambda]) = [r(\lambda)]$, $s([\lambda]) = [s(\lambda)]$ and $[\lambda][\mu] = [\lambda\mu]$. Define $\text{Per}(\Lambda) \subseteq \mathbb{Z}^k$ by

$$\text{Per}(\Lambda) := \{d(\mu) - d(\nu) : \mu, \nu \in \Lambda \text{ and } \mu \sim \nu\},$$

and define

$$H_{\text{Per}} := \{v \in \Lambda^0 : \text{for all } \lambda \in v\Lambda \text{ and } m \in \mathbb{N}^k \text{ such that } d(\lambda) - m \in \text{Per}(\Lambda), \text{ there exists } \mu \in v\Lambda^m \text{ such that } \lambda \sim \mu\}.$$  

For the next result recall that if $\Lambda$ is a $k$-graph then a subset $H$ of $\Lambda^0$ is hereditary if $s(H\Lambda) \subseteq H$.

**Theorem 4.2.** Let $\Lambda$ be a row-finite $k$-graph with no sources such that $\Lambda^0$ is a maximal tail.

1. The set $\text{Per}(\Lambda)$ is a subgroup of $\mathbb{Z}^k$.

2. The set $H_{\text{Per}}$ is a nonempty hereditary subset of $\Lambda^0$, and for all $p, q \in \mathbb{N}^k$ such that $p - q \in \text{Per}(\Lambda)$ and all $x \in H_{\text{Per}}\Lambda^\infty$, we have $\sigma^p(x) = \sigma^q(x)$.

3. If $r(\lambda) \in H_{\text{Per}}$ and $d(\lambda) - m \in \text{Per}(\Lambda)$, then there is a unique $\mu \in r(\lambda)\Lambda^m$ such that $\mu \sim \lambda$; in particular, if $\lambda \sim \mu$ and $d(\lambda) = d(\mu)$, then $\lambda = \mu$.

4. If $q_{\text{Per}} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k/\text{Per}(\Lambda)$ is the quotient map, then the set $\Gamma := (H_{\text{Per}}\Lambda)/\sim$ is a $q_{\text{Per}}(\mathbb{N}^k)$-graph with degree map $\overline{d} := q_{\text{Per}} \circ d$.

5. The assignment $\lambda \mapsto ([\lambda], d(\lambda))$ is an isomorphism of $H_{\text{Per}}\Lambda$ onto the pullback $q_{\text{Per}}^*\Gamma$.

**Proof of Theorem 4.2.** Since $\lambda \sim \lambda$ for all $\lambda \in \Lambda$, we have $0 \in \text{Per}(\Lambda)$. That $\sim$ is symmetric shows that $\text{Per}(\Lambda)$ is closed under inverses in $\mathbb{Z}^k$. To see that it is closed under addition, suppose that $m, n \in \text{Per}(\Lambda)$, and fix $\mu \sim \nu$ and $\eta \sim \zeta$ such that $d(\mu) - d(\nu) = m$ and $d(\eta) - d(\zeta) = n$. Then $d(\mu + \eta) - (m + n) = d(\mu) - m + d(\eta) - n = 0$, so $\mu + \eta \sim m + n$. Therefore, the assignment $\lambda \mapsto ([\lambda], d(\lambda))$ is well-defined. It is clear that $q_{\text{Per}}^*\Gamma$ is a $q_{\text{Per}}(\mathbb{N}^k)$-graph, and that $q_{\text{Per}}^*\Gamma$ is the pullback of $\Gamma$ over $H_{\text{Per}}\Lambda$. Finally, the assignment $\lambda \mapsto ([\lambda], d(\lambda))$ is an isomorphism of $H_{\text{Per}}\Lambda$ onto $q_{\text{Per}}^*\Gamma$, as desired.
and \(d(y) - d(\zeta) = n\). Since \(\Lambda^0\) is a maximal tail, there exist \(\alpha \in s(\mu)\Lambda \) and \(\beta \in s(\zeta)\Lambda\) such that \(s(\alpha) = s(\beta) = v\). We have \(\mu \alpha y = \nu \alpha y\) for all \(y \in v\Lambda^\infty\). Hence
\[
\sigma^{d(\mu)}(y) = \sigma^{d(\mu) + d(\nu \alpha)}(\nu \alpha y) = \sigma^{d(\nu) + d(\mu \alpha)}(\mu \alpha y) = \sigma^{d(\nu)}(y)
\]
for all \(y \in v\Lambda^\infty\); and similarly, \(\sigma^{d(\nu)}(y) = \sigma^{d(\zeta)}(y)\) for all \(y \in v\Lambda^\infty\). Now
\[
\sigma^{d(\mu) + d(\nu)}(y) = \sigma^{d(\mu)}(\sigma^{d(\nu)}(y)) = \sigma^{d(\mu)}(\sigma^{d(\zeta)}(y))
\]
\[
= \sigma^{d(\zeta)}(\sigma^{d(\mu)}(y)) = \sigma^{d(\zeta)}(\sigma^{d(\nu)}(y)) = \sigma^{d(\nu) + d(\zeta)}(y)
\]
for all \(y \in v\Lambda^\infty\). Fix \(\xi \in v\Lambda^{d(\mu) + d(\nu)}\) and \(\xi' \in s(\xi)\Lambda^{d(\nu) + d(\zeta)}\). Factorise \(\xi \xi' = \tau \tau'\) with \(d(\tau) = d(\xi')\) and \(d(\tau') = d(\xi)\). For \(z \in s(\xi')\Lambda^\infty\), we have \(\xi \xi' z \in v\Lambda^\infty\), and hence
\[
\xi' z = \sigma^{d(\zeta)}(\xi \xi' z) = \sigma^{d(\mu) + d(\nu)}(\xi \xi' z) = \sigma^{d(\nu) + d(\zeta)}(\tau \tau' z) = \tau' z.
\]
It follows that \(\xi' \sim \tau'\) and hence that \(m + n = d(\tau') - d(\xi')\) belongs to \(\text{Per}(\Lambda)\).

To prove Theorem 4.2, we need some technical results.

**Lemma 4.3.** Let \(\Lambda\) be a row-finite k-graph with no sources such that \(\Lambda^0\) is a maximal tail. For \(v \in \Lambda^0\), let \(\Sigma_v := \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : \sigma^p(x) = \sigma^q(x)\text{ for all }x \in v\Lambda^\infty\}\).

1. For each \(\lambda \in \Lambda\), we have \(\Sigma_{r(\lambda)} \subseteq \Sigma_{s(\lambda)}\).
2. The relation \(\Sigma_v\) is an equivalence relation on \(\mathbb{N}^k\) for each \(v \in \Lambda^0\).
3. The set \(\Sigma_v\) is a sub-monoid of \(\mathbb{N}^k \times \mathbb{N}^k\) for each \(v \in \Lambda^0\).

**Proof.**
1. Fix \(\lambda \in \Lambda\) and \((p,q) \in \Sigma_{r(\lambda)}\). For \(x \in s(\lambda)\Lambda^\infty\), we have
\[
\sigma^p(x) = \sigma^{d(\lambda)}(\sigma^p(\lambda x)) = \sigma^{d(\lambda)}(\sigma^q(\lambda x)) = \sigma^q(x),
\]
so \((p,q) \in \Sigma_{s(\lambda)}\).
2. It is routine to check that \(\Sigma_v\) is reflexive, symmetric and transitive.
3. Fix \(v \in \Lambda^0\). Part 2 implies that \((0,0) \in \Sigma_v\). Suppose that \((p,q), (p',q') \in \Sigma_v\), and fix \(x \in v\Lambda^\infty\). Then
\[
\sigma^{p+p'}(x) = \sigma^p(\sigma^{p'}(x)) = \sigma^p(\sigma^q(x)).
\]
Part 1 implies that \((p,q) \in \Sigma_{u(x,p,q)}\) = \(\Sigma_{r(s'(x))}\), and hence \(\sigma^p(\sigma^{q'}(x)) = \sigma^q(\sigma^{q'}(x)) = \sigma^{q+q'}(x)\). Hence \((p+p',q+q') \in \Sigma_v\).

**Proposition 4.4.** Let \(\Lambda\) be a row-finite k-graph with no sources such that \(\Lambda^0\) is a maximal tail. Let \(\Sigma_{\Lambda} = \bigcup_{v \in \Lambda^0} \Sigma_v\). Then \(\Sigma_{\Lambda}\) is an equivalence relation on \(\mathbb{N}^k\) and a submonoid of \(\mathbb{N}^k \times \mathbb{N}^k\). The set \(\Sigma_{\Lambda}^\text{min}\) of minimal elements of \(\Sigma_{\Lambda} \setminus \{0\}\) (under the usual ordering on \(\mathbb{N}^k \times \mathbb{N}^k\)) is finite and generates \(\Sigma_{\Lambda}\) as a monoid; and \((\Sigma_{\Lambda} \setminus \Sigma_{\Lambda}) \cap (\mathbb{N}^k \times \mathbb{N}^k) = \Sigma_{\Lambda}\).

**Proof.** That \(\Sigma_{\Lambda}\) is reflexive and symmetric follows from the same properties of the \(\Sigma_v\). If \((p,q), (q,r) \in \Sigma_{\Lambda}\) then there exist \(v, w\) such that \((p,q) \in \Sigma_v\) and \((q,r) \in \Sigma_w\). Since \(\Lambda^0\) is a maximal tail, there exists \(u \in \Lambda^0\) such that \(v\Lambda^u\) and \(w\Lambda^u\) are nonempty. Lemma 4.3(1) then implies that \((p,q), (q,r) \in \Sigma_u\). Lemma 4.3(2) therefore implies that \((p,r) \in \Sigma_u \subseteq \Sigma_{\Lambda}\).

We show that \(\Sigma_{\Lambda}\) is a submonoid of \(\mathbb{N}^k \times \mathbb{N}^k\). That each \(\Sigma_v\) is a monoid gives \((0,0) \in \Sigma_{\Lambda}\), so it suffices to show that \(\Sigma_{\Lambda}\) is closed under addition. Suppose that \((p,q)\) and \((r,s)\) belong to \(\Sigma_{\Lambda}\). As above, there exists \(u \in \Lambda^0\) such that \((p,q), (r,s) \in \Sigma_u\). Lemma 4.3(3) then implies that \((p,q) + (r,s) \in \Sigma_u \subseteq \Sigma_{\Lambda}\).

Let \((p,q) + (r,s) \in \Sigma_u \subseteq \mathbb{Z}^k \times \mathbb{Z}^k\). We must show that \(G \cap (\mathbb{N}^k \times \mathbb{N}^k) = \Sigma_{\Lambda}\). Since \(\Sigma_{\Lambda}\) is a monoid, the containment \(\supseteq\) is clear. For the reverse containment, suppose that \((p,q), (r,s) \in \Sigma_{\Lambda}\) and that \(p-r\) and \(q-s\) belong to \(\mathbb{N}^k\); we must show that \((p,q) - (r,s) \in \Sigma_{\Lambda}\). As above, there exists \(u \in \Lambda^0\) such that \((p,q)\) and \((r,s)\) both belong to \(\Sigma_u\). Fix
\[ \lambda \in u\Lambda^{r+s}, \text{ and let } w = s(\lambda). \text{ Lemma 4.3 implies that } (s, r) \text{ belongs to } \Sigma_u \text{ and then that } (p + s, q + r) \in \Sigma_u. \text{ So for } x \in w\Lambda^\infty, \text{ we have} \]
\[ \sigma^{p-r}(x) \sigma^{p+r}(\sigma^{s+r}(\lambda x)) = \sigma^{q-r}(\lambda x) = \sigma^{q+s}(\sigma^{r+s}(x)) = \sigma^{q-s}(\sigma^{r-s}(\lambda x)). \]

Thus \((p, q) - (r, s) \in \Sigma_w \subseteq \Sigma_A.\]

The set \(\Sigma_{\min}^A\) is completely unarranged in the sense that no two distinct elements of \(\Sigma_{\min}^A\) are comparable under \(\leq\). Hence \(\Sigma_{\min}^A\) is finite by Dickson’s Theorem (see [22, Theorem 5.1]). An induction (see, for example, [22, Proposition 8.1]) shows that it generates \(\Sigma_A\) as a monoid. \(\square\)

**Lemma 4.5.** Let \(\Lambda\) be a row-finite \(k\)-graph with no sources and suppose that \(\Lambda^0\) is a maximal tail. Suppose that \((p, q) \in \mathbb{N}^k\) and \(n \in \mathbb{Z}^k\) satisfy \((p - n, q - n) \in \Sigma_A\). Then \((p, q) \in \Sigma_A\).

**Proof.** Let \(n_+ = n \lor 0\) and \(n_- = (n) \lor 0\). Then \(n_+, n_- \in \mathbb{N}^k\) and \(n = n_+ - n_-\). Putting \(p' := p - n\) and \(q' := q - n\), we have
\[ (p, q) = ((p', q') + (n_+, n_+)) - (n_-, n_-) \in (\Sigma_A - \Sigma_A) \cap (\mathbb{N}^k \times \mathbb{N}^k). \]

Proposition 4.4 therefore implies that \((p, q) \in \Sigma_A.\) \(\square\)

**Lemma 4.6.** Let \(\Lambda\) be a row-finite \(k\)-graph with no sources such that \(\Lambda^0\) is a maximal tail. For \(p, q \in \mathbb{N}^k\), we have \((p, q) \in \Sigma_A\) if and only if \(p - q \in \text{Per}(\Lambda)\).

**Proof.** Suppose that \(p, q \in \mathbb{N}^k\) satisfy \(p - q \in \text{Per}(\Lambda)\). Fix \(\mu \sim \nu\) in \(\Lambda\) such that \(p - q = d(\mu) - d(\nu)\). Let \(p' = d(\mu)\) and \(q' = d(\nu)\). Since \(\mu \sim \nu\), for \(x \in s(\mu)\Lambda^\infty\) we have
\[ \sigma^{p'}(x) = \sigma^{p'+q'}(\nu x) = \sigma^{p'+q'}(\mu x) = \sigma^{q'}(x). \]

So \((p', q') \in \Sigma_{\min}(\mu) \subseteq \Sigma_A\). We have \(p - q = p' - q'\), and hence \(p - p' = q - q'\). Thus \(n := p - p' \in \mathbb{Z}^k\) and \(p, q \in \mathbb{N}^k\) satisfy \((p - n, q - n) \in \Sigma_A\). Thus Lemma 4.5 implies that \((p, q) \in \Sigma_A\).

Now suppose that \((p, q) \in \Sigma_A\). Fix \(v\) such that \((p, q) \in \Sigma_v\). Fix \(\lambda \in \nu\Lambda^{p+q}\). Factorise \(\lambda = \mu x = \nu \beta\) with \(d(\mu) = d(\beta) = p\) and \(d(\nu) = d(\alpha) = q\). For \(x \in s(\alpha)\Lambda^\infty\), we have
\[ \alpha x = \sigma^{\mu}(\mu x) = \sigma^{\nu}(\nu \beta x) = \beta x. \]

Hence \(\alpha \sim \beta\) and so \(p - q = d(\beta) - d(\alpha) \in \text{Per}(\Lambda)\). \(\square\)

**Proof of Theorem 4.2.** We begin by showing that \(H_{\text{Per}}\) is nonempty. By Proposition 4.4 the finite set \(\Sigma_{\min}^A\) generates \(\Sigma_A\) as a monoid. For each \((p, q) \in \Sigma_{\min}^A\) there exists \(w \in \Lambda^0\) such that \((p, q) \in \Sigma_w\). Let \(|\Sigma_{\min}^A|\) be the cardinality of \(\Sigma_{\min}^A\). By \(|\Sigma_{\min}^A|\) applications of condition 4.3 for the maximal tail \(\Lambda^0\), there exists \(w \in \Lambda^0\) such that \(\Sigma_{\min}^A \subseteq \Sigma_w\). Since \(\Sigma_w\) is a monoid and \(\Sigma_w \subseteq \Sigma_A\), it follows that \(\Sigma_w = \Sigma_A\). Let \(N := \vee\{p \lor q : (p, q) \in \Sigma_{\min}^A\}\), and fix \(\eta \in w\Lambda^N\). We claim that \(v := s(\eta)\) belongs to \(H_{\text{Per}}\). To see this, fix \(\lambda \in v\Lambda\) and \(m \in \mathbb{N}^k\) such that \(d(\lambda) - m \in \text{Per}(\Lambda)\). Lemma 4.6 implies that \((d(\lambda), m) \in \Sigma_A\). So Proposition 4.4 implies that there exist elements \((p_1, q_1), \ldots, (p_l, q_l) \in \Sigma_{\min}^A\) such that
\[ (d(\lambda), m) = \sum_{i=1}^{l} (p_i, q_i). \]

We will argue by induction that for \(0 \leq j \leq l\) there exists \(\lambda_j \in v\Lambda\) such that \(\lambda_j \sim \lambda\) and \(d(\lambda_j) = d(\lambda) + \sum_{i=1}^{j} (q_i - p_i)\). Putting \(\lambda_0 := \lambda\) establishes a base case. Now suppose that we have \(\lambda_{j-1}\) with the desired properties. Recall that \(r(\lambda) = s(\eta)\) where \(\eta \in \Lambda^N\) and \(\Sigma_{r(\eta)} = \Sigma_A\). Let \(\mu := \eta(N - q_j, N)\). By construction of \(\lambda_{j-1}\) we have
\[ d(\mu \lambda_{j-1}) = q_j + d(\lambda) + \sum_{i=1}^{j-1} (q_i - p_i) \geq p_j. \]
so $\lambda_j := (\mu \lambda_{j-1} + d(\mu \lambda_{j-1}))$ satisfies
\[ d(\lambda_j) = d(\mu \lambda_{j-1}) - p_j = d(\lambda) + \sum_{i=1}^{j} (q_i - p_i). \]

Fix $x \in s(\lambda_j)\Lambda^\infty$. We have
\[ \lambda_j x = (\mu \lambda_{j-1})(p_j, d(\mu \lambda_{j-1}))x = \sigma^{p_j}(\mu \lambda_{j-1} x). \]

Lemma 4.3 implies that $\Sigma_\lambda = \Sigma_{r(\eta)} \subseteq \Sigma_{r(\nu)} \subseteq \Sigma_\lambda$, giving equality throughout. Thus
\[ (p_j, q_j) \in \Sigma_{r(\nu)}, \] and so
\[ \sigma^{p_j}(\mu \lambda_{j-1} x) = \sigma^{q_j}(\mu \lambda_{j-1} x) = \lambda_{j-1} x. \]

So $\lambda_j \sim \lambda_{j-1}$. Since $\lambda_{j-1} \sim \lambda$ by the inductive hypothesis, $\lambda_j \sim \lambda$.

Now $\mu := \lambda_i$ satisfies $\mu \in \nu \Lambda^m$ and $\lambda \sim \mu$. Hence $\nu \in H_{\text{Per}}$ as claimed.

We show that $H_{\text{Per}}$ is hereditary. Suppose that $r(\alpha) \in H_{\text{Per}}$. Fix $\lambda \in s(\alpha) \Lambda$ and $m \in \mathbb{N}^k$ such that $d(\lambda) - m \in \text{Per}(\Lambda)$. Then $d(\alpha) - (m + d(\alpha)) \in \text{Per}(\Lambda)$. Hence there exists $\zeta \in r(\alpha)\Lambda$ such that $d(\zeta) = m + d(\alpha)$ and $\zeta \sim \alpha \lambda$. Since $\zeta x = \alpha \lambda x$ for all $x \in s(\lambda)\Lambda^\infty$, and since and $d(\zeta) > d(\alpha)$, we have $\zeta(0, d(\alpha)) = \alpha$. Thus $\zeta = \alpha \mu$ for some $\mu \in s(\alpha)\Lambda^m$.

For $x \in s(\lambda)\Lambda$,
\[ \lambda x = \sigma^{d(\alpha)}(\alpha \lambda x) = \sigma^{d(\alpha)}(\zeta x) = \sigma^{d(\alpha)}(\alpha \mu x) = \mu x. \]

So $\lambda \sim \mu$, whence $s(\alpha) \in H_{\text{Per}}$.

To see that $\sigma^q(x) = \sigma^q(xy)$ whenever $p - q \in \text{Per}(\Lambda)$ and $x \in H_{\text{Per}}\Lambda^\infty$, fix such $p, q, x$.

Let $x = (0, p)$. Then $x = \lambda \sigma^p(x)$. Since $r(x) \in H_{\text{Per}}$, there exists $\mu \in r(x)\Lambda^q$ such that $\mu \sim \lambda$. We then have
\[ \sigma^q(x) = \sigma^q(\mu \sigma^p(x)) = \sigma^q(\lambda \sigma^p(x)) = \sigma^q(x). \]

Proof of Theorem 4.2.1. Fix $\lambda \in H_{\text{Per}}\Lambda$ and $m \in \mathbb{N}^k$ such that $d(\lambda) - m \in \text{Per}(\Lambda)$. Suppose that $\mu, \nu \in \Lambda^m$ satisfy $\mu \sim \lambda \sim \nu$. Since $\sim$ is an equivalence relation, we have $\mu \sim \nu$. So it suffices to show that if $\mu \sim \nu$ and $d(\mu) = d(\nu)$, then $\mu = \nu$. Fix $x \in s(\mu)\Lambda^\infty$ and observe that $\mu \sim \nu$ implies $\mu x = \nu x$ and so $\mu = (\mu x)(0, m) = (\nu x)(0, m) = \nu$.

Proof of Theorem 4.2.4. We showed that $\Gamma$ is a category at the beginning of this section. Since $d$ is a functor, $d := q_{\text{Per}} \circ d$ is also a functor. We must show that $(\Gamma, d)$ has the unique factorisation property.

Let $q = q_{\text{Per}}(\Lambda) : \mathbb{Z}^k \rightarrow \mathbb{Z}^k / \text{Per}(\Lambda)$. Fix $\lambda \in \Lambda$ with $r(\lambda) \in H_{\text{Per}}$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = q(m) + q(n)$. We must show that there is a unique pair $\alpha, \beta \in \Gamma$ such that $d(\alpha) = q(m)$, $d(\beta) = q(n)$ and $\alpha \beta = [\lambda]$. We begin by showing that such a pair exists.

Since $d(\lambda) - (m + n) \in \text{Per}(\Lambda)$ and $r(\lambda) \in H_{\text{Per}}$, there exists $\mu \in r(\lambda)\Lambda^{m+n}$ such that $\mu \sim \lambda$. Let $\alpha := [\mu(0, m)]$ and $\beta := [\mu(m, m + n)]$. Then $\alpha \beta = [\mu(0, m)\mu(m, m + n)] = [\mu] = [\lambda]$ because $\mu \sim \lambda$; and $d(\alpha) = q(m)$ and $d(\beta) = q(n)$ by construction.

For uniqueness, suppose that $\mu_1, \nu_1, \mu_2, \nu_2 \in H_{\text{Per}}\Lambda$, satisfy $r(\nu_1) = s(\mu_1)$, $q(d(\mu_1)) = q(m)$ and $q(d(\nu_1)) = q(n)$. Suppose further that $\mu_1 \nu_1 \sim \lambda \sim \mu_2 \nu_2$. We must show that $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$. Fix $x \in s(\nu_1)\Lambda^\infty$. Then
\[ \nu_1 x = \sigma^{d(\mu_1)}(\mu_1 \nu_1 x) = \sigma^{d(\mu_1)}(\mu_2 \nu_2 x). \]

We have $q(d(\mu_1)) = q(m) = q(d(\mu_2))$, and so $d(\mu_1) - d(\mu_2) \in \text{Per}(\Lambda)$. Hence Theorem 4.2.1 implies that $\sigma^{d(\mu_1)}(y) = \sigma^{d(\mu_2)}(y)$ for all $y \in r(\mu_2)\Lambda^\infty$, and in particular
\[ \sigma^{d(\mu_1)}(\mu_2 \nu_2 x) = \sigma^{d(\mu_2)}(\mu_2 \nu_2 x) = \nu_2 x. \]

Thus $\nu_1 \sim \nu_2$. Since $d(\mu_1) - d(\mu_2) \in \text{Per}(\Lambda)$ and since $r(\mu_1) \in H_{\text{Per}}$, there exists a unique $\zeta \in r(\mu_1)\Lambda^{d(\mu_2)}$ such that $\mu_1 \sim \zeta$. Fix $x \in s(\lambda)\Lambda^\infty$. We have
\[ \zeta = (\zeta \nu_1 x)(0, d(\mu_2)) = (\mu_1 \nu_1 x)(0, d(\mu_2)) = (\mu_2 \nu_2 x)(0, d(\mu_2)) = \mu_2. \]
Proof of Theorem 4.2. It is routine to check that \( \lambda \mapsto ([\lambda], d(\lambda)) \) is a \( k \)-graph morphism from \( H_{\text{Per}} \Lambda \) to \( q_{\text{Per}(\Lambda)}^{*}(\Gamma) \). This \( k \)-graph morphism is injective by part (3). Fix \( \xi \in H_{\text{Per}} \Lambda \) and \( m \in \mathbb{N}^{k} \) such that \( m - d(\xi) \in \text{Per}(\Lambda) \). Then there exists \( \eta \in \rho(\xi)\Lambda_{m}^{*} \) such that \( \eta \sim \xi \). Thus \( ([\eta], d(\eta)) = ([\xi], m) \). So \( \lambda \mapsto ([\lambda], d(\lambda)) \) is surjective and hence an isomorphism. \( \square \)

### 5. The Primitive Ideals of a \( k \)-Graph Algebra

In this section we prove our main theorem, giving a complete listing of the primitive ideals in the \( C^{*} \)-algebra of a row-finite \( k \)-graph with no sources.

**Lemma 5.1.** Let \( \Lambda \) be a row-finite \( k \)-graph such that \( \Lambda^{0} \) is a maximal tail. Let \( H = H_{\text{Per}} \) be the hereditary set \([2]\), and let \( \Gamma = H\Lambda/\sim \). Let \( q = q_{\text{Per}} : \mathbb{Z}^{k} \to \mathbb{Z}^{k}/\text{Per}(\Lambda) \) be the quotient map, and let \( P := q(\mathbb{N}^{k}) \). Then

1. there is a unique \( * \)-homomorphism \( \phi : C^{*}(q^{\Gamma}) \to C^{*}(\Lambda) \) such that \( \phi(s_{([\lambda], d(\lambda))}) = s_{\lambda} \) for \( \lambda \in H\Lambda \);
2. the series \( \sum_{v \in H} s_{v} \) converges strictly to a projection \( P_{H} \) in \( M_{\text{Per}}^{*}(\Lambda) \), and \( \phi \) is an isomorphism of \( C^{*}(q^{\Gamma}) \) onto \( P_{H}C^{*}(\Lambda)P_{H} \); and
3. the corner \( P_{H}C^{*}(\Lambda)P_{H} \) is a full corner of the ideal \( I_{H} = \overline{\text{span}}\{s_{\lambda}s_{\mu}^{*} : \lambda, \mu \in \Lambda H\} \).

**Proof.** Theorem 4.2(3) implies that \( \lambda \mapsto ([\lambda], d(\lambda)) \) is a \( k \)-graph isomorphism between \( H\Lambda \) and \( f^{\Gamma} \). Statement (1) then follows from the universal property of \( C^{*}(q^{\Gamma}) \).

The argument of [3] Lemma 1.1 shows that \( \sum_{v \in H} s_{v} \) converges strictly to a multiplier projection \( P_{H} \) such that \( P_{H}s_{\mu}s_{\nu}^{*} = \chi_{H}(r(\mu))s_{\nu}s_{\mu}^{*} \). We have \( P_{H}C^{*}(\Lambda)P_{H} = \overline{\text{span}}\{s_{\lambda}s_{\mu}^{*} : \lambda, \mu \in \Lambda H\} \), so the image of \( \phi \) is \( P_{H}C^{*}(\Lambda)P_{H} \). Proposition 2.7 implies that \( \phi \) is injective, giving (2).

For (3), observe that \( I_{H} \) is the ideal generated by \( \{s_{v} : v \in H\} \) and so is generated as an ideal of \( C^{*}(\Lambda) \) by \( P_{H} \). Hence \( P_{H} \) determines a full corner in \( I_{H} \). \( \square \)

**Lemma 5.2.** Let \( \Lambda \) be a row-finite \( k \)-graph with no sources and let \( T \) be a maximal tail of \( \Lambda \). Then there is an infinite path \( x \in (\Lambda T)^{\infty} \) which is cofinal in \( T \) in the sense that for each \( v \in T \), there exists \( n \in \mathbb{N}^{k} \) such that \( vax(n) \neq \emptyset \).

**Proof.** Let \( \{v_{j}\}_{j=0}^{\infty} \) be a listing of \( T \). Condition (3) for a maximal tail implies that there exists \( \alpha_{1} \in v_{0}T \) such that \( v_{1}\Lambda s(\alpha_{1}) \neq \emptyset \). Applying the same condition again gives \( \alpha_{2} \in s(\alpha_{1})T \) such that \( v_{2}\Lambda s(\alpha_{2}) \neq \emptyset \). Inductively, we construct \( \{\alpha_{j}\}_{j=1}^{\infty} \) such that \( r(\alpha_{j}) = s(\alpha_{j-1}) \) and \( v_{j}\Lambda s(\alpha_{j}) \neq \emptyset \) for all \( j \). Define \( \beta_{0} = v_{0} \) and \( \beta_{i} = \beta_{i-1}\alpha_{i} \) for \( i \geq 1 \). Then \( v_{j}\Lambda s(\beta_{j}) \neq \emptyset \) whenever \( i \leq j \), and \( d(\beta_{j}) \to (\infty, \ldots, \infty) \) as \( j \to \infty \). The argument of [12, Remarks 2.2] shows that there is a unique \( x \in (\Lambda T)^{\infty} \) such that \( x(0, d(\beta_{j})) = \beta_{j} \) for all \( i \). This \( x \) is cofinal by construction. \( \square \)

In the following, given an infinite path \( x \) in a \( k \)-graph \( \Lambda \), we write \([x]\) for the shift-tail equivalence class \( \{\lambda\sigma^{m}(x) : m \in \mathbb{N}^{k}, \lambda \in \Lambda x(m)\} \).

**Theorem 5.3.** Let \( \Lambda \) be a row-finite \( k \)-graph with no sources.

1. Let \( T \) be a maximal tail of \( \Lambda \) and \( \gamma \) a character of \( \text{Per}(\Lambda T) \). Suppose that \( z \in \mathbb{T}^{k} \) satisfies \( z^{m-n} = \gamma(m - n) \) for all \( m - n \in \text{Per}(\Lambda T) \). Suppose that \( x \in (\Lambda T)^{\infty} \) is cofinal in \( T \). Then there is an irreducible representation \( \pi_{[x],z} : C^{*}(\Lambda) \to B(\ell^{2}([x])) \) determined by

\[
\pi_{[x],z}(s_{\lambda}) = \begin{cases} 
z^{d(\lambda)}\xi_{\lambda y} & \text{if } y(0) = s(\lambda) \\
0 & \text{otherwise.}
\end{cases}
\]
(2) Let $I$ be a primitive ideal in $C^*(\Lambda)$. Then $T = \{v \in \Lambda^0 : s_v \notin I\}$ is a maximal tail. Let $H = H_{\text{Per}}(\Lambda T)$. There is a unique character $\gamma$ of $\text{Per}(\Lambda T)$ such that $s_\mu - \gamma(d(\mu) - d(\nu))s_\nu \in I$ whenever $\mu, \nu \in \text{HAT}$ and $\mu \sim_{\Lambda T} \nu$. Suppose that $x \in (\Lambda T)^\infty$ is cofinal in $\Lambda T$ and that $z \in \mathbb{T}^k$ satisfies $z^{m-n} = \gamma(m-n)$ for all $m-n \in \text{Per}(\Lambda T)$. Then $I = \ker \pi_{[x],z}$.

Proof. (1) It is routine to check that the operators $\{\pi_{[x],z}(s_\lambda) : \lambda \in \Lambda\}$ defined by (3) constitute a Cuntz-Krieger $\Lambda$-family. The universal property of $C^*(\Lambda)$ then yields a homomorphism $\pi := \pi_{[x],z} : C^*(\Lambda) \to B(\ell^2([x]))$ extending (3).

Fix $y \in [x]$. For $w \in [x]$ and $n \in \mathbb{N}^k$, 

$$\pi(s_{y(n,0)}s_{y(0,n)})\delta_w = \begin{cases} \delta_w & \text{if } w(0,n) = y(0,n) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\lim_{n \to \infty} \pi(s_{y(n,0)}s_{y(0,n)})\delta_w = \theta_{y,\delta_w} \delta_w$. Thus the net $(\pi(s_{y(n,0)}s_{y(0,n)})\delta_w)_{n \in \mathbb{N}^k}$ converges strongly to $\theta_{y,y}$. We claim that the strong closure of the image of $\pi$ contains $K(\ell^2([x]))$. Fix $w, y \in [x]$ and fix $p_i, q_i \in \mathbb{N}^k$ such that $\sigma^{p_i}(w) = \sigma^{q_i}(x)$ and $\sigma^{p_2}(y) = \sigma^{q_2}(x)$. Let $q = q_1 \lor q_2$, $\alpha = w(0, p_1 + (q - q_1))$ and $\beta = y(0, p_2 + (q - q_2))$. Then $w = \alpha\sigma(y)$ and $y = \beta\sigma(y)$. We have $\pi(s_{x(i,0)\sigma(y)+n})\pi(s_{x(i,0)\sigma(y)+n}) = \pi(s_{x(i,0)\sigma(y)+n})\pi(s_{x(i,0)\sigma(y)+n})$ which converges strongly to $\pi(s_{x(i,0)\sigma(y)+n})\pi(s_{x(i,0)\sigma(y)+n})$ as claimed. Hence $\pi$ is irreducible.

(2) Since $I$ is a primitive ideal in $C^*(\Lambda)$, there is an irreducible representation $\pi : C^*(\Lambda) \to B(H)$ such that $\ker \pi = I$. Let $K = \{v \in \Lambda^0 : \pi(s_v) = 0\}$. Then [17, Theorem 5.2(a)] implies that $K$ is hereditary and saturated. So the set 

$$T = \{v \in \Lambda^0 : s_v \notin I\} = \{v \in \Lambda^0 : \pi(s_v) \neq 0\} = \Lambda^0 \setminus K$$

satisfies conditions (1) and (3) of a maximal tail. To establish (3), fix $v, w \in T$ and $h \in p_i\mathcal{H}\setminus\{0\}$. Since $\pi$ is irreducible, $h$ is cyclic for $\pi$. So there exists $a \in C^*(\Lambda)$ such that $\pi(a)h \in \pi(p_i)\mathcal{H}\setminus\{0\}$; that is, $0 \neq \pi(p_i)\pi(a)h = \pi(p_iap_i)h$. In particular, $p_iap_i \neq 0$. Since $C^*(\Lambda) = \overline{\text{span}}(s_{a,s_{a}} : s(\alpha) = s(\beta))$ we may approximate $a$ by a linear combination of such $s_{a,s_{a}}$. Thus there exist $\alpha, \beta$ with $s(\alpha) = s(\beta)$ such that $p_iap_i \neq 0$. Hence $u := s(\alpha) = s(\beta)$ satisfies $u \mathcal{H}, u \mathcal{H} \neq \theta$, and so $T$ satisfies (3).

Let $H = H_{\text{Per}}(\Lambda T)$, and let $I_{\overline{H}}$ be the ideal of $C^*(\Lambda)$ generated by $\{s_v : v \in H\}$. Since $\pi|_{I_{\overline{H}}}$ is nonzero, Theorem 1.3.4 of [1] shows that $\pi|_{I_{\overline{H}}} := \pi|_{I_{\overline{H}}}$ is an irreducible representation. Lemma 5.1[2] implies that $I_{\overline{H}}P_H$ is a Morita equivalence between $P_HP^*(\Lambda)P_H$, so $\ker(\pi_H)$ is induced from the primitive ideal ker$(\pi_H)|_{P_HC^*(\Lambda)P_H}$. Let $\Gamma = (\text{HAT})/\sim$. Then Lemma 5.1[2] provides an isomorphism $\omega : C^*(\sigma^\Gamma) \to P_HC^*(\Lambda)P_H$, and then $\pi_H \omega \sigma$ is an irreducible representation $C^*(\sigma^\Gamma)$. Now Theorem 3.3 implies (see Remark 3.6) that there is a unique character $\gamma$ of $\text{Per}(\Lambda T)$ such that $s_\mu - \gamma(d(\mu) - d(\nu))s_\nu \in I$ whenever $\mu, \nu \in \text{HAT}$ and $\mu \sim_{\Lambda T} \nu$. It is routine to check that if $x$ is cofinal in $\Lambda T$ and $z^{m-n} = \gamma(m-n)$ for $m-n \in \text{Per}(\Lambda T)$, then $s_\mu - \gamma(d(\mu) - d(\nu))s_\nu \in \ker \pi_{[x],z}$ whenever $\mu, \nu \in \text{HAT}$ and $\mu \sim_{\Lambda T} \nu$.

We claim that $\ker(\pi_H|_{P_HI_{\overline{H}}P_H}) = \ker(\pi_{[x],z}|_{P_HI_{\overline{H}}P_H})$. Each of $\pi \circ \omega$ and $\pi_{[x],z} \circ \omega$ is a representation of $C^*(\sigma^\Gamma)$ which vanishes on the generators of the ideal $I_z$ of Lemma 3.4 and is nonzero on every vertex projection $s_{(v,0)} \in C^*(\sigma^\Gamma)$. So $\pi \omega \sigma$ and $\pi_{[x],z} \circ \omega$ descend to representations $\rho$ and $\rho_{[x],z} \circ \omega$ of $C^*(\sigma^\Gamma)/I_z$. Lemma 3.4 gives an isomorphism $\theta : C^*(\Gamma) \to C^*(\sigma^\Gamma)/I_z$, and the representations $\rho \circ \theta$ and $\rho_{[x],z} \circ \theta$ of $C^*(\Gamma)$ are nonzero on vertex projections. Hence Corollary 2.8 implies that $\rho \circ \theta$ and $\rho_{[x],z} \circ \theta$ are both injective. Thus $\rho$ and $\rho_{[x],z}$ are both injective, which implies that ker$(\pi \circ \omega) = I_z = \ker(\pi_{[x],z} \circ \omega)$. Since
$P_H$ is a strict limit of projections in $I_H$, we have $P_H I_H P_H = P_H C^*(\Lambda) P_H$, and so $\omega$ is an isomorphism from $C^*(q^* \Gamma)$ to $P_H I_H P_H$. This proves the claim.

Since $I_H P_H$ is a Morita equivalence, the claim implies that $\ker(\pi_H) = \ker(\pi_{[x],z} |_{I_H})$. Now [19, Proposition 2.72] applied to the adjointable left action of $C^*(\Lambda)$ by left multiplication on the standard Hilbert module $(I_H)_{I_H}$ implies that $\pi = \ker(\pi_{[x],z})$. $\square$

Let $T$ be a maximal tail of $\Lambda$ and $\gamma$ a character of $\text{Per}(\Lambda T)$. Theorem 5.3 implies that if $x, y$ are both cofinal in $T$ and $w, z \in \mathbb{T}^k$ satisfy $w^{m-n} = \gamma(m-n) = z^{m-n}$ whenever $m-n \in \text{Per}(\Lambda T)$, then $\ker(\pi_{[x],z}) = \ker(\pi_{[y],w})$. We define $I_{T,\gamma} := \ker(\pi_{[x],z})$. We write $\text{MT}(\Lambda)$ for the collection of all maximal tails in $\Lambda^0$.

**Corollary 5.4.** Let $\Lambda$ be a row finite $k$-graph with no sources. The assignment $(T, \gamma) \mapsto I_{T,\gamma}$ is a bijection from $\bigcup_{T \in \text{MT}(\Lambda)} \{T\} \times \text{Per}(\Lambda T)$ to the set of primitive ideals of $C^*(\Lambda)$.

**Proof.** This follows immediately from Theorem 5.3 and Lemma 5.2. $\square$

**Remark 5.5.** It is worthwhile to point out what the above catalogue of primitive ideals says for aperiodic maximal tails $T$. In this instance, we have $\text{Per}(\Lambda T) = \{0\}$ so the only character is the identity character $\gamma(0) = 1$. The Cuntz-Krieger uniqueness theorem implies that $\pi_{[x],1}$ restricts to a faithful representation of $C^*(H\Lambda T)$. Thus $I_{T,1}$ is precisely the gauge-invariant primitive ideal associated to $T$ described in [11]. In particular, if $\Lambda$ is strongly aperiodic in the sense of [11], so that every maximal tail $T$ is aperiodic, then our result recovers the listing given in [11].

We close by characterising primitivity of $C^*(\Lambda)$; recall that a $C^*$-algebra $A$ is primitive if it has a faithful irreducible representation.

**Corollary 5.6.** Let $\Lambda$ be a row finite $k$-graph with no sources. Then $C^*(\Lambda)$ is primitive if and only if $\Lambda^0$ is a maximal tail and $\Lambda$ is aperiodic.

**Proof.** First suppose that $\Lambda^0$ is a maximal tail and $\Lambda$ is aperiodic. Lemma 5.2 implies that there is an infinite path $x$ of $\Lambda$ which is cofinal in $\Lambda^0$. Consider the irreducible representation $\pi_{[x],1}$ of $\Lambda$. Since $\Lambda$ is a maximal tail, we have $\pi_{[x],1}(s_v) \neq 0$ for all $v \in \Lambda^0$. Since $\Lambda$ is aperiodic, the Cuntz-Krieger uniqueness theorem implies that $\pi_{[x],1}$ is faithful. Hence $C^*(\Lambda)$ is primitive.

Now suppose that $C^*(\Lambda)$ is primitive. Then $\{0\}$ is a primitive ideal of $C^*(\Lambda)$. Theorem 5.3 implies that $T := \{v \in \Lambda^0 : s_v \notin \{0\}\}$ is a maximal tail, and that there is a character $\chi$ of $\text{Per}(T)$ such that $I_{T,\chi} = \{0\}$. Since the generators of $C^*(\Lambda)$ are all nonzero, $T = \Lambda^0$, so $\Lambda^0$ is a maximal tail. To see that $\Lambda$ is aperiodic, we must show that $\text{Per}(\Lambda) = \{0\}$. By Theorem 4.2, it suffices to show that $\mu \sim \nu$ implies $d(\mu) = d(\nu)$. Suppose that $\mu \sim \nu$. Then $s_\mu - \chi(d(\mu) - d(\nu))s_\nu \in I_{T,\chi}$, and hence $s_\mu = \chi(d(\mu) - d(\nu))s_\nu$. For $z \in \mathbb{T}^k$, we then have

$$z^{d(\mu)} s_\mu = \gamma_z(s_\mu) = \gamma_z(\chi(d(\mu) - d(\nu))s_\nu) = z^{d(\nu)} \chi(d(\mu) - d(\nu))s_\nu.$$ 

So $z^{d(\mu) - d(\nu)} s_\mu = \chi(d(\mu) - d(\nu))s_\nu$ for all $z \in \mathbb{T}^k$. Since the right-hand side is nonzero and independent of $z$, we deduce that $z^{d(\mu) - d(\nu)}$ is constant with respect to $z$, forcing $d(\mu) = d(\nu)$. $\square$

**References**


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