

## REVIEWS

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*Randomness and Recurrence in Dynamical Systems.* By Rodney Nillsen. The Carus Mathematical Monographs Number 31, Mathematical Association of America, Washington, DC, 2010, xviii + 357 pp., ISBN 978-0-88385-043-5, \$52.95.

*Reviewed by James A. Walsh*

What do you teach in your Department's first undergraduate real analysis course? Is the content of the course independent of the instructor? Do you stick to the real line, or work in  $n$ -dimensional Euclidean space, teach metric spaces, or perhaps go so far as to introduce a bit of measure theory?

Our first analysis course, until recently called "advanced calculus," is an introduction to the real line and functions defined thereon. When I teach the course I typically begin by reminding students what continuous and differentiable functions are, all in the context of limits, working very carefully with  $\epsilon$ - $\delta$  proofs throughout. I prove the Mean Value Theorem in week three via the Extreme Value Theorem, even though the proof of the latter comes later in the semester. I take a quick tour through the theory of the Riemann integral for bounded functions, introducing the ideas of supremum and infimum. I prove the Fundamental Theorem of Calculus assuming the Intermediate Value Theorem, whose proof is also delayed until later in the course. Including time spent on sequences of real numbers, this represents a bit more than the first third of the course.

I have taken this approach to afford students a first go at developing at least some facility with  $\epsilon$ - $\delta$  proofs and "the art of the inequality," while simultaneously reminding them of the big theorems of single variable calculus. Moreover, I emphasize that a much more in-depth understanding of the set of real numbers is needed to prove, for example, the Intermediate and Extreme Value Theorems, and the existence of infima and suprema for non-empty bounded sets. I like to think that at this point the stage is set for the heart of the course: A thorough treatment of the completeness of the reals.

As in many first real analysis courses, topics such as the axiom of continuity (as in [4]), convergence of bounded monotonic sequences, density of the rationals and irrationals, nested interval theorems, and Cauchy sequences are presented. We then move on to point-set theory: limit points, open, closed, connected, compact sets (including a proof of the Heine-Borel Theorem). The Intermediate Value Theorem is proved in the process of showing that continuous functions map connected sets to connected sets, while the Extreme Value Theorem follows from our proof that continuous functions map compact sets to compact sets. Final course topics include uniform continuity (with a proof that continuous functions are Riemann integrable), and pointwise and uniform convergence of sequences of functions. If a class or two remains at the end, I might introduce metric spaces.

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Does this sound at all like your course? Our analysis course is often the first upper-level course taken by our majors, one generally considered to be among our most challenging offerings. Students are expected to continue to learn the art of writing clear and concise proofs, while simultaneously coming to terms with the challenge (and beauty!) inherent in the study of the real number line. With a bit of personal bias, surely, I like to think my course is a hefty, solid introduction to the subject, one with which I am for the most part pleased.

As is the case with my course, however, I wonder if you too have students questioning the need for such a high level of care, precision and hard work when the payoff is a sequence of proofs of theorems with which they are already familiar and which they see as more or less obvious. It would benefit such students to see introductory analysis ideas appearing in novel settings, bearing in mind that it cannot simply be all flash as students still need to solve problems and prove theorems along the way.

With this in mind, I at one time introduced the study of the dynamics of circle homeomorphisms into my analysis course, a topic which connects quite nicely with ideas and techniques in the course [5]. It was not clear to me, however, that students found this as interesting as I did. A second difficulty was the additional level of abstraction presented by the notion of a "lift" of a circle map to the real line. (A third difficulty, of course, was finding space for this material in the course, about which I will comment later in this review.)

Rodney Nillsen has written a wonderful book centered on his stated goal of making the introductory analysis course more relevant. In particular, Nillsen strove to connect basic analysis to probability theory, and to certain aspects of chaotic dynamical systems, all the while assuming minimal prior knowledge. He has succeeded most admirably.

A note, first, on Nillsen's writing style. He states in the Foreword:

In some ways the work should be regarded as "lecture notes," in that questions of motivation of the ideas, and thinking about the results from different viewpoints, are considered perhaps more than usual. There is a certain emphasis on how we might think "metaphorically" about possible meanings of concepts and results, and not only about the formal aspects of the results. Occasionally, there is mention of connections to other areas of inquiry.

The motivation for the theorems presented, as well as for the execution of their proofs, is extraordinarily well done. In addition, all proofs are complete and carefully written. He also mentions the need for and pays heed to precise mathematical statements throughout the book. In short, this is just the type of book you want your analysis student to pick up as it has been successfully written with just such a student in mind.

In terms of content, the author is essentially interested in seeing how far he can push elementary analysis to prove major theorems from ergodic theory, without any use of general measure theory save the notion of measure zero sets. (Birkhoff's Ergodic Theorem, in particular, does not come into play until the end of the book.) Thus, distributions of sequences and orbits of certain discrete dynamical systems forge the connection to probability theory. Most everything takes place in the interval  $[0, 1]$ , so that the length of a subinterval  $A$  can be associated with the probability of the occurrence of the event  $A$ . (Measure zero sets then correspond to probability zero events.) The study of sequences and orbits of discrete dynamical systems in the interval  $[0, 1]$  also lends itself nicely to introductory analysis material.

Following a brief introductory chapter, sequences generated by iteration of the function

$$\tau_\alpha : [0, 1) \rightarrow [0, 1), \quad x \mapsto \langle x + \alpha \rangle,$$

where  $\alpha$  is constant and  $\langle x + \alpha \rangle$  means  $(x + \alpha) \bmod 1$ , are studied in chapter 2. (I should note that Nillsen does not initially place the study of the distribution of sequences in the dynamical systems context.) Given  $x$  in  $[0, 1)$ , set  $x_0 = x$  and, for  $n \geq 1$ , set  $x_n = \tau_\alpha(x_{n-1})$ . The sequence  $\{x_n\}$  is the  $\tau_\alpha$ -orbit of  $x$ ; note that  $x_n = \tau_\alpha^n(x_0)$ , where  $\tau_\alpha^n$  represents  $n$ -fold composition of  $\tau_\alpha$  with itself. Kronecker's Theorem states that, for  $\alpha$  irrational, the  $\tau_\alpha$ -orbit of zero, namely  $\langle \alpha \rangle, \langle 2\alpha \rangle, \langle 3\alpha \rangle, \dots$ , is dense in  $[0, 1)$ . The fact the  $\tau_\alpha$ -orbit of any  $x$  in  $[0, 1)$  is dense in  $[0, 1)$  is stated, with a request for a proof placed in the chapter exercises.

A common proof of Kronecker's Theorem invokes the compactness of the circle of circumference one, along with the Bolzano-Weierstrass Theorem. True to one of his stated goals, Nillsen uses nothing more than the pigeon-hole principle. As is the case throughout the book, all statements are proved completely and carefully, clearly written with the student in mind.

Weyl's Theorem concerns the distribution of the infinite sequence  $\{\langle m\alpha \rangle\}$  in  $[0, 1)$  for  $\alpha$  irrational. If  $\mu(J)$  denotes the length of a subinterval  $J$  of  $[0, 1)$ , it states that the proportion of the first  $n$  terms in  $\{\langle m\alpha \rangle\}$  that lie in  $J$  approaches  $\mu(J)$  as  $n \rightarrow \infty$ . Though the proof is necessarily significantly longer due to the avoidance of all things measure-theoretic, it lies within the grasp of the introductory analysis student.

An historically noteworthy problem concerns the frequency of digits appearing in the base- $b$  expansion of a number in  $[0, 1)$ . Nillsen introduces sets of measure zero, and independent sets/events, in chapter 3. He first uses the notion of independence to prove a *recurrence* result: Every finite string of 0s and 1s occurs infinitely often in the base-2 expansion of almost every  $x$  in  $[0, 1)$ . Rademacher functions are then introduced and used as the major tool in proving Borel's Theorem: Almost every  $x$  in  $[0, 1)$  is *simply normal* to the base 2 (see also [2]). A number is simply normal to the base 2 if the proportion of 1s in its first  $n$  binary digits tends to  $1/2$  as  $n \rightarrow \infty$ .

What of the frequency of finite strings  $r_1 r_2 \dots r_s$  of 0s and 1s? A number  $x$  in  $[0, 1)$  is *normal* to the base 2 if the proportion of the number of occurrences of each finite string of 0s and 1s of length  $s$  in its first  $n$  binary digits tends to  $1/2^s$  as  $n \rightarrow \infty$ . Nillsen proves the Normal Numbers Theorem, i.e., that almost every  $x$  in  $[0, 1)$  is normal to the base 2, using certain products of Rademacher functions (called *Walsh* functions) as the key tool. Though measure theory is avoided, the price here is somewhat steep as the material is technical and notationally cumbersome (the author does warn the reader that "this investigation lies deeper than earlier results").

As a final topic in the third chapter, Nillsen considers the distribution of the leading digits of geometric sequences of integers. He provides a proof of the fact that such digits obey *Benford's Law*: The frequency of a first digit  $d \in \{1, 2, \dots, 9\}$  in such a sequence of numbers is, interestingly,  $\log_{10}(1 + 1/d)$  (see also [3]).

The penultimate chapter concerns "length-preserving" transformations, which Nillsen is able to intuitively connect to a system in which the probability of an event is independent of the past history of the system. Length here means the usual length—for an interval  $[a, b)$ ,  $\mu([a, b)) = b - a$ —though it is extended to a collection of subintervals of a given bounded interval, called *basic sets* by the author, which form an algebra of sets. Basic sets are finite unions of pairwise disjoint subintervals which are viewed from the probabilistic perspective as the events that can legitimately occur. A lovely proof of Poincaré's Recurrence Theorem is provided in this setting: If  $U$  is

a basic subset of positive length in a bounded interval  $S$ , and if  $f : S \rightarrow S$  preserves length, then for almost all  $x$  in  $U$ ,  $f^n(x)$  is in  $U$  for some  $n$  (and in fact  $f^n(x)$  is in  $U$  for infinitely many  $n$ ).

With Poincaré's theorem in hand, attention turns to Kac's Theorem on average return times. Suppose  $f : S \rightarrow S$  and  $U$  are given as in the preceding paragraph. How long does it take, on average, for points in  $U$  to return to  $U$  under iteration of  $f$ ? Kac's Theorem, which generally is stated in the measure theory setting with an invariant *ergodic* measure  $\mu$ , avers that this average return time is the ratio  $\mu(S)/\mu(U)$ . Nillsen avoids the ergodic hypothesis by assuming a weaker hypothesis which, it turns out, suffices in the more restrictive setting of length-preserving transformations and the algebra of basic sets.

With each of these results in the first four chapters, Nillsen returns to his metaphor relating the randomness or frequency of strings of digits in the base-2 expansion of a number with the repeated flipping of an unbiased coin (heads might be 0, tails 1). This again points to his desire to ensure that the reader stays with him and to reconnect with the probabilistic perspective permeating the book.

The final chapter provides the reader with a taste of the power of Birkhoff's Ergodic Theorem. This chapter is more expository than those preceding it as more advanced ideas are presented. Not all details are given here, not all statements are proved. In particular, the invariant measures in most examples are not shown to be ergodic. Once again, however, Nillsen writes with a remarkable clarity as he introduces the basics of measure theory. The motivation for and discussion of Birkhoff's theorem is wonderful, relating "time averages" to "space averages." Nillsen uses this result to revisit the theorems of Weyl and Borel (now working with base- $b$  expansions,  $b \geq 2$  an integer), as well as the Normal Numbers Theorem and Kac's Theorem on average return times. This is perfect motivation for a student to go off and immerse herself in a text on measure theory.

Nillsen concludes by pointing to ways in which Birkhoff's theorem can be applied to more general expansions, including Lüroth and continued fraction expansions. There is a nice treatment of a result pertaining to the frequency of digits appearing in the continued fraction expansion of a number in  $[0, 1)$ . Indeed, this book segues cleanly into another Carus Monograph, *Ergodic Theory of Numbers* by Dajani and Kraaikamp [1], which is more advanced in perspective, with measure theory and Birkhoff's theorem front and center from the start.

The material in chapter 5 provides for an ideal coda to this work. Throughout previous chapters, Nillsen used many non-technical euphemisms for an ergodic dynamical system, including random, mixing, unpredictable, unbiased, one exhibiting information loss. The curtain is pulled back in the final chapter to connect all of these descriptions with their intended meaning in this monograph, that of an invariant ergodic measure.

Each chapter of *Randomness and Recurrence in Dynamical Systems* begins with a remarkably thorough flow chart to guide the reader through the subsequent pages. Each contains a *Notes* section, which is informative and often highly entertaining. It is in this venue that Nillsen tackles such topics as mathematical reasoning, proof, and the philosophical nature of mathematics, the philosophical foundations of probability theory, connections between randomness, algorithmic complexity and Turing machines, and Poincaré recurrence vis-à-vis statistical mechanics. Each chapter ends with a collection of mostly-challenging exercises, as well as a set of more open-ended "investigations." The average number of references per chapter is a robust 26.

The reader has perhaps been given the impression this is the perfect book? Typos do exist, though they are minimal and do not interfere with one's understanding of

the text. The third chapter is exceedingly long, with the material on Walsh functions (connected to the non-measure-theoretic proof of the Normal Numbers Theorem) and Benford's Law fairly difficult to slog through. The treatment of rational numbers and their decimal expansions could be shortened (or moved in part to the exercises). I was also surprised by the length of figure captions, which at times extended in such a way as to make it difficult to determine where the caption ended and where the text continued. Be sure to read all captions, however, as they are informative and, on occasion, contain definitions that are used later in the book.

The larger issue is this: If a good student is to work hard going through this material, why not have her learn measure theory from the get-go? One answer might be that a course in measure theory is offered infrequently, if at all, in your Department (we offer one every few years at Oberlin). More importantly, however, I'd argue that Nillsen has managed to present a wealth of classical and beautiful mathematics at a level appropriate for your introductory analysis student. I can envision using this book to enrich our sophomore-level proofs course (discrete mathematics), as well as my analysis course.

What form might this enrichment take in the analysis course? I essentially incorporated topics from the dynamics of circle homeomorphisms into my course via exercises, sprinkled in throughout the semester, with little class time devoted to this material. That is, the circle map material was added to the existing course. Granted, this took place earlier in my career, at a time when I appear to have pushed students a bit harder than I do presently (I wonder if mid- and late-career readers feel similarly about their teaching?). The last several times I have taught analysis I have omitted the material on circle maps.

Nillsen's book has forced me to revisit the issue of revamping my analysis course, at least in part, by incorporating more of the utility of analysis techniques and results, particularly as they arise in novel, interesting settings. Beyond the enhancement of the course in this way, this work might also be used as a text for a seminar-type course, or for use with students in private reading courses. In any setting, *Randomness and Recurrence in Dynamical Systems* is a gem of a book, a welcome and worthy addition to the Carus Mathematical Monograph series.

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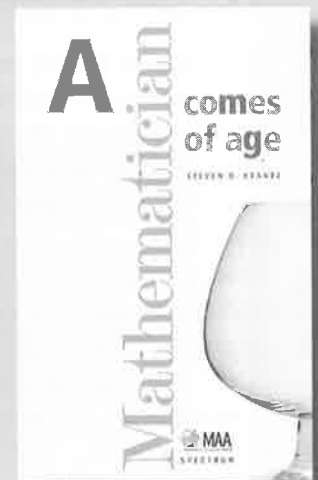
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