

Igor Kluváněk

His life, achievements and influence in Australia

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In Memoriam Igor Kluváněk (1931–1993)

Abstract

Igor Kluváněk arrived in Australia in 1967, and he took up a position in mathematics at the Flinders University of South Australia. As well as his major research achievements, he also had a commitment to mathematical exposition, at both the advanced and elementary levels. He had a large number of research students. In this paper are discussed his mathematical achievements, ideas and influence while he was in Australia. Igor Kluváněk also had serious intellectual interests and commitments outside mathematics, and parts even of his mathematical research are scattered with philosophical analysis, ideas on education, aphorisms, humorous comments and *bons mots*. He studied at St Paul's seminary in Kensington, Sydney in the early 1980s and later on at St Dominic's in Camberwell, Melbourne. He resigned from his professorship at Flinders in 1986. However, he continued his mathematical research, spending time in the Centre for Mathematical Analysis at the Australian National University, and also at Macquarie University. He produced further substantial writings from these periods of mathematical, theological and philosophical study, and these also are discussed. Igor Kluváněk's mathematical researches in Australia emphasised integration theory in a very broad sense, and vector measures in particular. This paper is written for a general audience, and also for those who may have a mathematical background but not be familiar with the areas of Igor Kluváněk's researches. However, the abstract mathematical concepts and research mentioned in parts of this paper will be introduced through consideration of two of his more expository articles, on the ideas of Archimedes concerning the calculation of areas and their relevance to modern mathematical analysis.

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1 Introduction

Igor Kluvánek was a close friend of mine over a long period, and naturally I will have cause to refer to him often in this paper. So, I will refer to him mostly as “Igor”. I will endeavour to do some sort of justice to all of those activities and achievements of Igor’s with which I have some familiarity. Let me say that my knowledge of Igor’s many-faceted life is incomplete, so there may be things I mention of which members of this audience, or people elsewhere, may be better informed than I am. I should also comment that in this paper, those aspects of Igor’s mathematical work that I emphasise are influenced, to an extent, by how easy or difficult it is to explain those ideas to a mixed audience. This also affects the overall exposition in this paper. So, any undue emphasis there might be in my discussion should not be taken to reflect some sort of judgment on the relative importance of the various areas of Igor’s research or of his achievements in those areas. Suffice it to say that in all the areas of Igor’s research, he made very important and even crucial contributions, and inspired and influenced much new mathematics, also through his students and colleagues. But Igor was also more than his mathematics, and that is one of the reasons he was found by so many to be an inspirational figure, even within mathematics. Biographical details about Igor may be found in the article on him in *Wikipedia*, in the introduction to [15], and in [62]. In this paper, I will discuss his mathematical achievements, his exposition, his ideas on pedagogy, and his philosophical and religious ideas and writings. I will also try and convey something of Igor’s character, personality and modes of thought, which also contributed to making him so distinctive and memorable a person.

2 Archimedes and the notion of integration

In this paper, I must inevitably talk about Igor’s mathematics, in its scientific seriousness and its implications. A common theme in Igor’s research was the mathematical process of *integration*. As I am trying to make these ideas as accessible as possible to a varied audience, I will first give an idea of the integration process as deriving from two of Igor’s expository papers *Archimedes was right* I and II [35, 36], published in 1987. As well as setting a background for his research achievements, these papers illustrate an important aspect of Igor’s thought. That is, his looking at and reconsidering even the most fundamental concepts in mathematics, however well-known and understood they were considered to be, in order to see how well we really understand them, to see them in their simplest possible form, and to derive new ideas and make progress from critical re-examination of them. The papers also illustrate his historical awareness, not just that type of awareness of history as a body of facts about the past, but as a repository of ideas for the present.

Archimedes considered the problem of calculating the area of a parabolic section, and he used two different methods to do so. Igor was interested in the second, less well-known method, and argued in these papers that this method may be useful as a modern way of introducing mathematical integration — that is to say, in classical terms, the problem of calculating areas. As he said [35, page 51]:

... I wish to demonstrate that the ideas stemming from the second method [of Archimedes] can stream-line the presentation of the Lebesgue integral in the class-room to the extent that, by comparison, the Riemann integral would appear just a cumbersome museum exhibit. They can also contribute to our understanding of the “inner working” of integration theory and thereby open it to new possibilities of development and applications.

Figures 1 to 3 briefly indicate this method of Archimedes, as described by Igor. The idea is to successively approximate the area of the parabolic section by adding up the areas of a sequence of triangles that will eventually “exhaust” the area we are interested in. This works well for the specific problem considered by Archimedes but, as Igor pointed out, the method depends heavily upon the particular properties of the parabola, and does not provide a method for calculating an area having a more general curve as its boundary. It was Igor’s aim to describe such a method, deriving from Archimedes’ method for the parabola. His approach is explained and illustrated in Figures 4 to 6. This also serves to introduce the general idea of integration. Here, the concept of a *function* is important, even crucial. Note that the term *integration* of a curve or function refers to the fact that the integration process, that is the calculation of the area under the pictorial representation (or graph) of the function, depends on *all* the function values, and so “integrates” them to give a single number. In Figure 5, the graph of a function f is illustrated. The problem is to calculate the shaded area under the graph between the numbers a and b . This area is variously denoted, but let’s say that since the function is f , since we denote the usual notion of length by λ , and since we are integrating the area under the graph of f between a and b , this area is denoted by

$$\int_{[a,b]} f d\lambda,$$

and is called the *integral of f over the interval $[a, b]$ with respect to λ* .

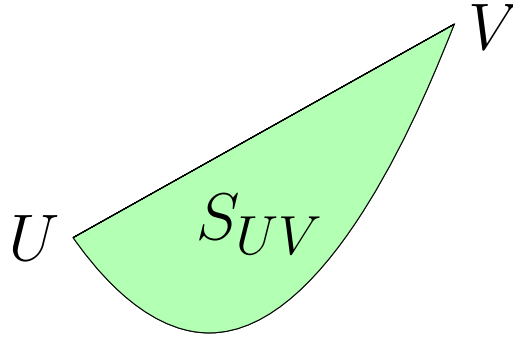


Figure 1. The shaded region in the figure illustrates a parabolic section. It is called the parabolic section determined by U and V , and is denoted by S_{UV} . The aim is to calculate the area of S_{UV} .

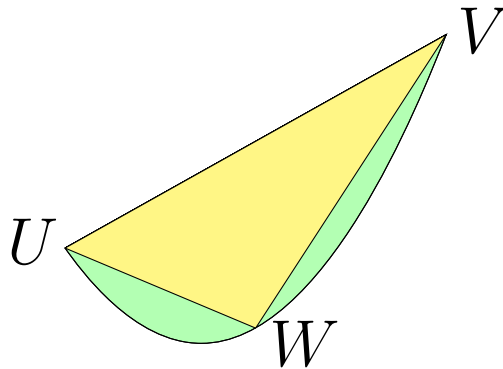


Figure 2. In the figure, W is vertically below the midpoint of the line segment UV . We obtain a triangle UVW , whose area approximates the area of the parabolic section.

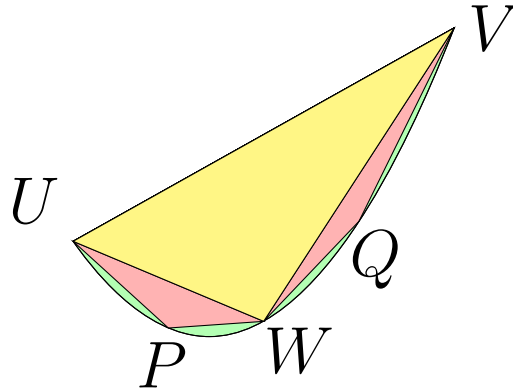


Figure 3. The procedure we carried out for the line segment UV , we now carry out again for the line segments UW and VW . We obtain two new triangles, UWP and VWQ . The *sum* of the areas of the three triangles we have constructed gives an approximation to the area of the parabolic section S_{UV} . This procedure is continued indefinitely, so that the (infinity of) triangles so obtained “exhaust” the parabolic section. Archimedes was able to calculate the sum of the areas of all these triangles, deducing that the area of the parabolic section is $4/3$ of the area of the original triangle UVW .

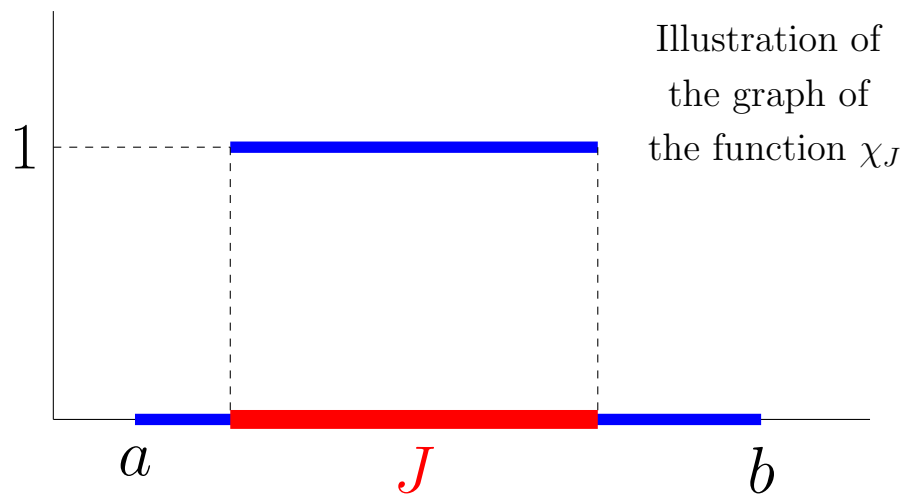


Figure 4. The figure illustrates a subinterval J of a given interval $[a, b]$. The function χ_J takes the value 1 at each point of J , and the value 0 elsewhere.

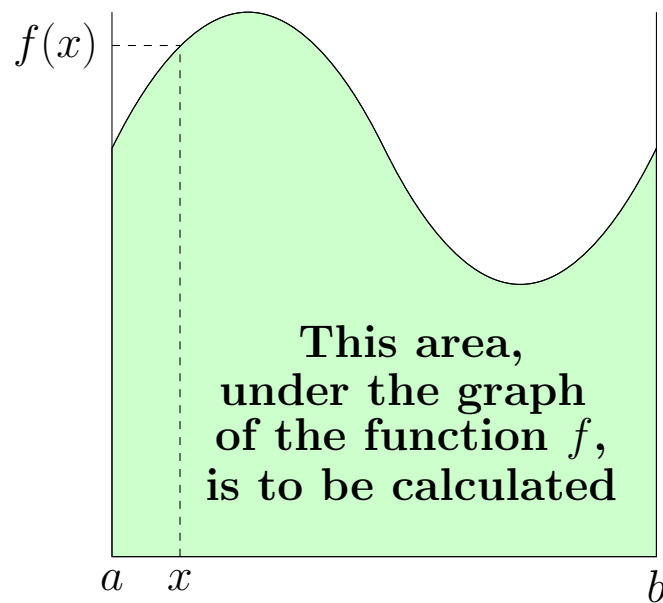


Figure 5. The interval $[a, b]$ of numbers is given, as shown. For each point x in $[a, b]$, the function assigns a real number, $f(x)$. Letting $\mathbb{R}$ denote the set of all real numbers, we may write this as $f : [a, b] \rightarrow \mathbb{R}$. Whereas Archimedes calculated the area under a parabolic graph, the aim is to provide a technique to calculate the area under an arbitrary graph.

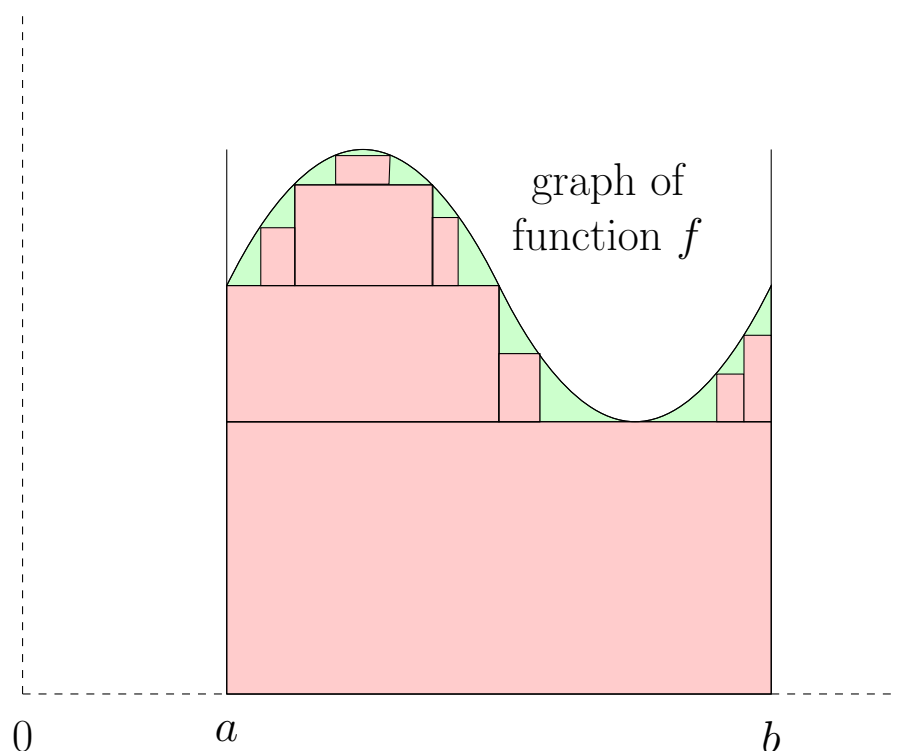


Figure 6. The area under the graph of f is approximated by the sum of the areas of the shaded rectangles. In the figure there is only a finite number of rectangles, but as we add more and more rectangles we will get a better and better approximation so that, as we continue indefinitely, the actual area $\int_{[a,b]} f \, d\lambda$ is the sum of the areas of an infinite number of rectangles.

Figure 6 illustrates Igor’s approach, deriving from the second approach of Archimedes to calculating the area of a parabolic section, and also from a calculation of Nicole d’Oresme in the 1350s, who showed that

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots = 2. \quad (2.1)$$

The idea is to “exhaust” the area by means of adding more and more rectangles, and then taking the sum of the areas of all the rectangles, of which there will be, in general, an infinite number. But Igor was not simply interested in applying this idea of Archimedes to the classical problem of calculating areas. Although it is not fully evident in these two papers, he was interested in this approach in more general and abstract contexts, and in widening the scope of integration theory. There will be more comment about this later, but let me introduce some pertinent ideas now.

First, there is the notion of a summable sequence. If (u_n) is a sequence of non-negative numbers we say that the *sum* of the sequence (u_n) is the smallest number s , if it exists, such that

$$u_1 + u_2 + \cdots + u_n \leq s, \text{ for all } n = 1, 2, 3, \dots,$$

and we say that (u_n) is *summable* with *sum* s . This is written as

$$u_1 + u_2 + u_3 + \cdots = s.$$

Similarly, Igor introduces the notion of a summable sequence and its sum for sequences of negative numbers. If the sequence (u_n) consists of a mixture of negative and non-negative numbers, the sequence is summable if the subsequences of negative and non-negative terms are each summable, and the sum of (u_n) is

$$(b_1 + b_2 + b_3 + \cdots) + (c_1 + c_2 + c_3 + \cdots),$$

where (b_n) is the subsequence of non-negative terms and (c_n) is the subsequence of negative terms. In particular, Oresme's result in (2.1) says in this context that the sequence

$$\frac{1}{2}, \frac{2}{2^2}, \frac{3}{2^3}, \frac{4}{2^4}, \dots$$

is summable with sum 2. Also, if a, b are numbers with $a \leq b$, we denote by $[a, b]$ the set of all numbers x with $a \leq x \leq b$, and $[a, b]$ is called a (*closed*) *interval*. Then, the *length* $\lambda(J)$ of an interval J is denoted by $\lambda(J)$. If I is an interval and J is subinterval of I , the function χ_J is given by

$$\chi_J(x) = \begin{cases} 0, & \text{if } x \text{ is not in } J, \\ 1, & \text{if } x \text{ is in } J. \end{cases}$$

(See Figure 4, for these concepts.) The definition of χ_J means that there is a correspondence between sets and functions taking on only the values 0 and 1 — the set J corresponds to χ_J and vice-versa. This correspondence is used quite frequently in Igor's research (see [28, page 22], for example).

Igor then said in [36, page 88] that a function f on the interval $[a, b]$, as in Figure 5 or 6, is *Archimedes integrable* if there is a sequence (J_n) of sub-intervals of $[a, b]$ and numbers $c_1, c_2, c_3, \dots$ such that the following (2.2) and (2.3) hold. The sequence

$$c_1\lambda(J_1), c_2\lambda(J_2), c_3\lambda(J_3), \dots \text{ is summable.} \quad (2.2)$$

If x is in I and the sequence

$$c_1\chi_{J_1}(x), c_2\chi_{J_2}(x), c_3\chi_{J_3}(x), \dots$$

is summable, then

$$f(x) = c_1\chi_{J_1}(x) + c_2\chi_{J_2}(x) + c_3\chi_{J_3}(x) + \cdots \quad (2.3)$$

Then, the (Archimedes) integral $\int_{[a,b]} f d\lambda$ of f over $[a, b]$ is the sum

$$c_1\lambda(J_1) + c_2\lambda(J_2) + c_3\lambda(J_3) + \cdots \quad (2.4)$$

In terms of Figure 6, the idea behind (2.2), (2.3) and (2.4) is that the integral $\int_{[a,b]} f d\lambda$ is the sum of the areas of all rectangles under the graph of f , assuming we have an infinite sequence of rectangles that “exhaust” the area under the graph (note that the figure only depicts a finite number of these rectangles).

Igor comments in [36, page 90] that he is not introducing a new integral, but rather a specific method for introducing the Lebesgue integral, and refers to the books of J. Mikusiński [54] and of F. Riesz and B. Sz-Nagy [64]. A reason for mentioning the “Archimedes definition” of the integral is that both prior to and after these papers it influenced Igor’s own work and, for example, that of his students Brian Jefferies and Susumu Okada (see [17, 26, 43, 57]).

3 Mathematical research

3.1 General remarks

Now that some general notion of integration has been introduced, I will try and describe a general context which covers much of Igor’s research work. It is of course this work which contributed, in no small part, to making him such a well-recognised mathematician in Australia, and internationally. We may have a certain mathematical or scientific procedure, say some (possibly generalised) notion of integration. We apply this procedure to a certain class (or set) of objects, thus obtaining a new class of objects. One question is: what properties do these new objects have? A related question is: what is it that *characterises* these new objects individually, in a way that is independent of the original procedure by which they have been obtained? That is, we are trying to perceive these new objects as arising in two different ways. A third question is: what properties do these new objects have *collectively*, as a class? A related question is: what *characterises* these new objects collectively, as a class? In this case, we are trying to understand the *type* of the new class that we obtain when we apply the procedure to a given class. A final question is: once we have answered, or at the least have clarified or partially answered such questions, what implications, such as applications, arise from them?

A great deal of Igor’s mathematical work falls into clarifying and answering such questions, in particular and even mainly, when the procedure is integration, taken in a broad meaning and in many different contexts. I will discuss several aspects of his work from this viewpoint, and they mostly, but not all, involve the area in which Igor was a pioneer and seminal thinker — namely, vector measures.

What is a vector measure? It is a generalised notion of length which, by definition, has certain properties one naturally would expect length to have. The notion is generalised in that, whereas the area $\mu(J)$ of an interval J of real numbers is a *number*, in the case where μ is a vector measure, the “length” or “area” $\mu(J)$ of a set J is in a finite or infinite-dimensional space, usually, say, a *Banach space*¹. If the Banach space is denoted by X , the constituents of X are called *vectors* and there is a notion of length in X , so that every vector x in X has a length, denoted by $\|x\|$. The sets J which are allowed to have a “length” come from

¹More generally, this may be a locally convex topological vector space. That is, a type of set of “vectors” which may be added together and also multiplied by numbers, a structure (the topology) that enables one to talk about continuity, and this structure is determined by a family of convex sets (see Figure 7).

a family of subsets of some set S , let's denote it by Σ , which has certain properties which allow us to call Σ a σ -algebra of subsets of S . Thus, the vector measure μ is a function,

$$\mu : \Sigma \longrightarrow X,$$

which assigns to every set A in Σ a vector $\mu(A)$ in X .² A vector measure satisfies certain other properties which we expect a notion of length to have. Just as we integrated a function $f : S \longrightarrow \mathbb{R}$ as in Figure 6, using the vector measure λ corresponding to ordinary length to obtain a number $\int_S f d\lambda$ in $\mathbb{R}$, we may be able to integrate a function $f : S \longrightarrow \mathbb{R}$ which is compatible or *measurable* with respect to Σ , to obtain a vector $\int_S f d\mu$ in X .

Igor was one of the pioneers in the theory of vector measures. One of his contributions was before his coming to Australia, but it is of such importance that I will mention it briefly. This work, in [20], was his vector measure version of the extension theorem of C. Carathéodory, a key fact in classical integration theory. In one of the main works on vector measures, the book by J. Diestel and J. J. Uhl [5, page 37], concerning this work the authors say:

The extension theorem for vector measures as it appears here is a major contribution of I. Kluváněk to the theory of vector measures . . . When one recalls that a theory of strongly additive measures was not available at that time, one must be very impressed by Kluváněk's results.

Another important result of Igor's before he came to Australia was what has become known widely as Kluváněk's Sampling Theorem, in harmonic analysis [21]. The importance of this work can be gauged from the review paper [13] and the review of Igor's work at the beginning of [15]. A converse of Igor's sampling theorem has been discussed recently in [1]. However, nearly all of Igor's work in Australia was to be in vector measures and related areas.

3.2 The range of a vector measure

Since a vector measure is a function $\mu : \Sigma \longrightarrow X$, we can ask: what properties does the set $\{\mu(A) : A \text{ is in } \Sigma\}$ have? That is, what can we say about the range of μ ? In the case of ordinary length, we take S to be some interval³ $[a, b]$ of numbers, and then we can consider all the subintervals of S , and take μ to be the ordinary length λ . Then, $[a, b]$ has length $b - a$, every subinterval has length at most $b - a$, every point has length 0, and so we see that

$$\text{The range of } \mu = \left\{ \lambda(J) : J \text{ is a subinterval of } [a, b] \right\} = [0, b - a].$$

Consequently, we see that the range of λ is an interval of numbers and so is a convex set — that is, every line segment joining any two points of the set lies within the set (see Figure 7). It is also *compact* in the sense that an infinite sequence of points in $[a, b]$ must “accumulate”

²If (A_n) is a sequence of sets, $\cup_{n=1}^{\infty} A_n$ is the set consisting of those elements that are in any single one, or more, of the sets in the sequence (A_n) . The *empty set* has no elements, and is denoted by $\emptyset$. A function $\mu : \Sigma \longrightarrow X$, as above, is a vector measure if it satisfies the following condition: whenever (A_n) is a sequence of sets in Σ such that $A_m \cap A_n = \emptyset$ whenever $m \neq n$, we have $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

³See Figure 4.

at some point in $[a, b]$.⁴ Now, a famous result of Liapunov [51] is that if we have a non-atomic vector measure⁵ on a σ -algebra of sets, that takes its value in a *finite-dimensional* space, then the range of the measure is compact and convex, just as it is in the case of ordinary length. However, it was known that the result of Liapunov need not apply when the vector measure takes its values in an infinite-dimensional space. Igor's papers [25, 29] clarified what happens to the range of a vector measure in the infinite-dimensional case. Here is one of his theorems from [25] (actually this combines two theorems but neither is used in its full generality).

Theorem. *Let S be a set, let Σ be a σ -algebra of subsets on S , let X be a Banach space and let $\mu : \Sigma \longrightarrow X$ be a non-atomic vector measure. Then the following sets (i), (ii) and (iii) are all equal:*

- (i) *The closed convex set generated by the range of μ .*
- (ii) *The weak closure of the range of μ .*
- (iii) *The set*

$$\left\{ \int_S f d\mu : f \text{ is measurable and } f : S \longrightarrow [0, 1] \right\}.$$

Also, the (equal) sets in (i), (ii) and (iii) are weakly compact.

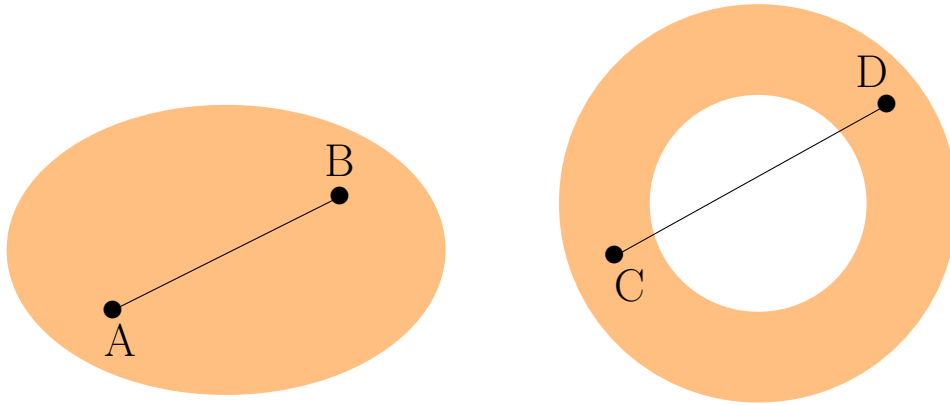


Figure 7. The set on the left (indicated by the shaded area) is *convex*, in that for any points A, B in the set, the line segment joining A and B lies in the set. The set on the right is not convex, because the line segment joining the points C, D in the set does not lie entirely within the set.

Note that (i) and (ii) together tell us that the weak closure of the range of μ is convex. Note that when X is a locally convex space rather than a Banach space, the above result may not hold. But it will hold if X has certain other properties and if the vector measure is

⁴In finite dimensions, a set is *compact* if it is closed (in that it includes any boundary points) and bounded (in that its points cannot get away from the origin by an arbitrarily large distance). We may regard compactness of a set as a property substituting for finiteness of the set.

⁵Here, “non-atomic” means that there is no set A in Σ such that for some B in Σ it happens that B is contained by A but $\mu(B)$ is neither 0 nor $\mu(A)$. It is a type of “continuity” condition on μ .

closed. The concept of a closed vector measure was a significant advance, and was introduced by Igor in [25].

3.3 Control theory and vector measures

Now, consider a fundamental control problem. A car has a maximum acceleration of α_0 , and a maximum deceleration of β_0 . The maximum possible speed of the car is v_0 . The problem is: if the car is stationary when the time is zero, how should the car accelerate and decelerate to cover a given distance s in minimum time, becoming stationary after having covered the distance s ? Here is an idea of the solution.

One control strategy is to accelerate at the maximum acceleration until a certain speed is attained, and then decelerate at the maximum deceleration until the car stops, while also making sure the car covers the exact distance s . This is an example of what is called *bang-bang control*, in that only two extremes are used in the control strategy; namely maximum acceleration and then maximum deceleration. Other strategies are possible, say having a more complex mixture of acceleration and deceleration, not necessarily at the maximum values; but it turns out that it is bang-bang control as described that minimises the time it takes to cover the given distance. In general, time optimal control is where a control function is to be found so that the solution of the problem meets a continuously moving target in minimum possible time. For example, as pointed out in [12, page 43], making a rendezvous with an orbiting satellite is a problem of this type, as is the problem of stabilising growth of a population (say) in minimal time.

A *bang-bang principle* in control theory is one that gives conditions under which, if a minimum time can be achieved by an admissible control, then it can be achieved by an admissible bang-bang control. The situation is described by Hermes and Lasalle [12, page 46] as follows:

It had been an intuitive assumption for some time that if the control for a system is operating from a limited source of power and if it is desired to have the system change from one state to another in minimum time, then it is necessary at all times to utilize all the power available; that is, to use bang-bang control. The intuitive feeling is that, if full power is not being used, the use of additional power available can always speed up the process.

Now, it may not be clear what all this has to do with Igor's work on the range of a vector measure but, in fact, in 1959 Lasalle [50] had used Liapunov's Theorem to derive sufficient conditions for a bang-bang principle to apply. It meant that for controls taking values in a given compact, convex subset U of real n -dimensional space $\mathbb{R}^n$, any point that could be reached by such a control could also be reached by a control taking values only from the extreme points of U .⁶ Admissible controls that take their values only from the extreme points correspond to bang-bang controls, and this gives the connection between functional analysis, vector measures and bang-bang principles. This approach to control theory in finite dimensions was systematically developed by Hermes and Lasalle [12]. Owing to this rôle of Liapunov's Theorem in finite-dimensional space, and because Igor's work on the range of

⁶An extreme point of a set is one which does not lie in the interior of any line segment joining two points in the set (see Figure 9).

vector measures had clarified what happened to Liapunov's result in infinite dimensions, it was natural to investigate control theory for the infinite-dimensional case. Important work was carried out by Igor and his student, Greg Knowles, in [26, 27, 28]. Their book [28] entitled "Vector measures and Control Systems" aroused wide interest, and some of its ideas are now discussed.

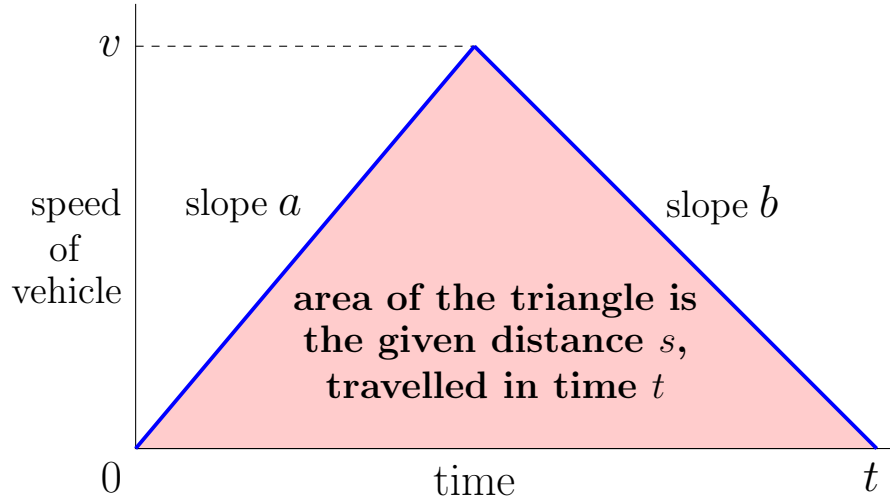


Figure 8. The figure illustrates a control problem. A vehicle is to travel a given distance s , starting and finishing at rest. In the figure, time is on the horizontal axis, speed is on the vertical axis. The vehicle has a constant acceleration a followed by a constant deceleration b , represented by the slopes of the sides of the triangle, and a maximum speed of v is reached. The area of the triangle is the distance covered, so this must equal s . We have to pick the acceleration a and the deceleration b so as to minimise the time t taken, subject to the above conditions. But the time taken is the length of the base of the triangle, and to minimise this length while keeping the area of the triangle the same, equal to s , it seems that we need to make the sides of the triangle as steep as possible. That is, to minimise the time it appears we should use the maximum possible acceleration and the maximum possible deceleration. Now suppose the maximum possible acceleration is a_0 and the maximum possible deceleration is b_0 . We regard deceleration as a negative acceleration, so that $a_0 > 0$ and $b_0 < 0$. Thus, the control problem is to choose a, b from the interval $[b_0, a_0]$, so as to minimise the time, and we have seen that the way to do this is to choose the *end-points* a_0, b_0 from the interval of all possible accelerations. These end-points are also called *extreme* points. Thus, to solve the control problem we use the *extreme controls*, or the *bang-bang controls*, from the set of all available controls. Igor Kluváněk's work in control theory was concerned with developing ideas like these in the infinite-dimensional context.

In finite dimensions, a control system might be described by an equation of the form

$$x'(t) = A(t)x(t) + u(t), \text{ for } t \geq 0. \quad (3.1)$$

Here, t denotes time, u is the control in the system, and x is the function (the solution of the equation) that describes what happens when the control u is used for the system. The

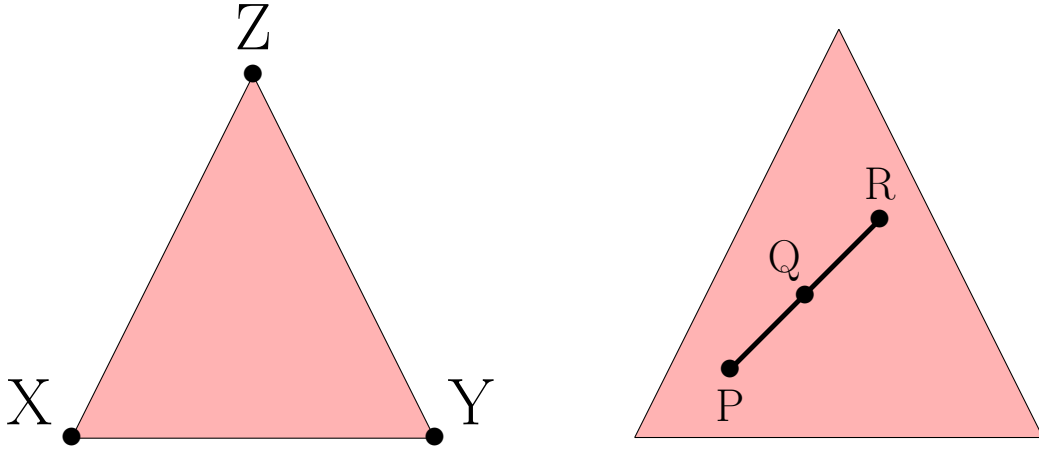


Figure 9. The shaded triangular region XYZ , on the left, is convex. Its extreme points are the vertices X, Y, Z . No other point is extreme, for every other point lies in the interior of a line segment which lies within the triangular region. On the right, Q is not an extreme point for the region because it is interior to the line segment PR , which lies entirely within the region.

functions $x, u : \mathbb{R} \longrightarrow \mathbb{R}^n$ and so, x and u are *vector valued* functions. The term x' is the rate of change of the solution x , and $A(t)$ is a square matrix. Equation (3.1) expresses the idea that the rate of change of the solution x at time t equals combinations of the coordinates of x at time t plus the coordinates of the control u at time t . Furthermore, we want the solution at time 0 to be some prescribed value x_0 — that is, we want $x(0) = x_0$. The value of this solution of equation (3.1) at time t , given the control u , we denote by $x(t, u)$. Typically, the controls u we can use will be constrained, by a condition that for some compact subset U of $\mathbb{R}^n$, we must have $u(t)$ is in U for all times t . Such controls will be called *admissible*. If the target is at $z(t)$ at time t , the possibility of meeting the target at time t by using an admissible control is equivalent to saying that $z(t)$ is in the *admissible set*

$$\mathcal{A}(t) = \left\{ x(t, u) : u \text{ is an admissible control} \right\}.$$

For, in this case there is an admissible control $\tilde{u}$ such that

$$x(t, \tilde{u}) = z(t),$$

and we see that meeting the target occurs at time t when we use the control $\tilde{u}$. The *reachable set* $\mathcal{R}(t)$ at time t of the control system, is essentially the attainable set of a simpler system equivalent to the original system. The sets $\mathcal{A}(t)$ and $\mathcal{R}(t)$ are related by an equation of the form

$$\mathcal{A}(t) = X(t)[x_0 + \mathcal{R}(t)],$$

where X is an invertible-matrix valued function associated with the problem. This means that properties about $\mathcal{A}(t)$ can be obtained from $\mathcal{R}(t)$ and vice-versa. The point about $\mathcal{R}(t)$ is that it is related to Lyapunov's Theorem because of the equation

$$\mathcal{R}(t) = \left\{ \int_0^t X^{-1}(\tau)u(\tau) d\tau : u \text{ is admissible} \right\}. \quad (3.2)$$

For, if the admissible controls $u = (u_1, \dots, u_n)$ are those obtained by requiring that $|u_j(\tau)| \leq 1$ for all $0 \leq \tau \leq t$, we can deduce from Liapunov's Theorem [12, Corollary 8.2, page 25] that

$$\begin{aligned} & \left\{ \int_0^t X^{-1}(\tau)u(\tau) d\tau : |u_j(\tau)| = 1, \text{ for all } t \text{ and } 1 \leq j \leq n \right\} \\ &= \left\{ \int_0^t X^{-1}(\tau)u(\tau) d\tau : u \text{ is admissible} \right\}. \end{aligned} \quad (3.3)$$

We deduce from (3.2) and (3.3) that

$$\mathcal{R}(t) = \left\{ \int_0^t X^{-1}(\tau)u(\tau) d\tau : |u_j(\tau)| = 1, \text{ for all } t \text{ and } 1 \leq j \leq n \right\}. \quad (3.4)$$

But the controls on the right hand side of (3.4) are the bang-bang controls only, so we see that (3.4) establishes a bang-bang principle.

Now, Igor and Greg Knowles noted the following in [28, page 98]:

... in infinite-dimensional control systems the bang-bang principle need not hold. This is an exact parallel to the fact that Lyapunov's Theorem need not hold for infinite-dimensional non-atomic vector measures.

Their response was to introduce the notion of *Liapunov measures* and *non-Liapunov measures* [26, 28]. The Liapunov measures lead, in infinite dimensions, to an equality like that in (3.4) above, establishing a connection in infinite dimensions between vector measures and bang-bang principles. In so doing they provided a new proof of Liapunov's Theorem. They showed that a closed vector measure could be decomposed into Liapunov and non-Liapunov components, and that a closed vector measure always has a Liapunov extension.

3.4 Spectral theory

In mathematical physics, many problems can be described by means of an *integral equation*. If K denotes a real-valued function, $K : \mathbb{R}^2 \rightarrow \mathbb{R}$, and if a, b are given real numbers with $a < b$, the equation may take the form

$$\int_a^b K(x, y)f(y) dy = g(x), \text{ for } a \leq x \leq b. \quad (3.5)$$

The idea in (3.5) is that the left hand side changes the function $f : [a, b] \rightarrow \mathbb{R}$ into the function $g : [a, b] \rightarrow \mathbb{R}$. Following Igor's dictum, that a mathematical object may be denoted by a single symbol, we may write this equation simply as

$$A(f) = g.$$

Then, A stands for the *process* of integrating a function f using the “kernel” K , to obtain a function g , in accordance with the formula (3.5). We call A an *operator*. A typical general setting is where we have a Banach space X and $A : X \longrightarrow X$ is linear and bounded⁷ say, a set Ω with a σ -algebra $\mathcal{S}$, and an additive and multiplicative function $P : \mathcal{S} \longrightarrow B(X)$, where $B(X)$ denotes the space of bounded operators on X . Such a function may be called a *spectral measure*. It is a special type of vector measure. Whereas the finite-dimensional case is classical, where it is concerned with diagonalization of matrices, and the (infinite-dimensional) Hilbert space case was also well-known, spectral measures in more general settings became prominent with the work of N. Dunford. As Igor wrote in [30, page 108]:

The conscious use of vector measures by N. Dunford led to a considerable extension of applicability of spectral theory and to the introduction of the concept of a spectral operator in Banach or, more generally, locally convex spaces.

With a spectral measure is (possibly) associated a process of integration which, when applied to a real or complex-valued function, produces an operator A in $B(X)$. Interest then arises in the operators that can arise in this way. This is due, in part, to historical and scientific reasons, since physical phenomena have been, and still are, frequently understood in terms of integral equations, which lead to the more abstract idea of an operator. An operator A in $B(X)$ is called *spectral*, or is said to be of *scalar type*, if there is a spectral measure P such that the procedure of integration by P produces the given operator. This is written as

$$A = \int \lambda P(d\lambda).$$

Igor’s student, Werner Ricker, carried out extensive work on spectral theory. Commenting on some of this work, Igor wrote [33, page 35]:

The theory of extended spectral operators, due to W. Ricker, is not yet available in a monograph form. Its point of departure is the observation that the failure of an operator ... to be of scalar type may be, so to say, not the fault of the operator itself ... but, rather, of the space...

Igor also published on spectral theory and other aspects of vector measures in [24, 31, 33, 42, 47]. Werner Ricker’s early work on the product of spectral measures and extended spectral operators appeared in [59, 60], but he went on to do more work on spectral theory, including the book [61], which used both vector measures and spectral theory, citing Igor’s own work on spectral theory. In [33, 47] Igor, together with Werner Ricker in the second of these papers, discussed further notions of a scalar-type operator, and related ideas. In this work, the Archimedes approach to integration, as enunciated in [35, 36], played a central rôle, but now in an advanced research context, in line with Igor’s view that the ideas of Archimedes are, in a sense, still relevant today.

⁷Here, A is linear means that it respects the properties of integration in the sense that $A(f + g) = A(f) + A(g)$ for all f, g in X and $A(af) = aA(f)$ for all numbers a and f in X . In this context, boundedness of A is the same as continuity.

3.5 The scope of integration theory

An early paper reflecting the “Archimedes” approach to integration in an abstract setting is by Susumu Okada [57], who defined a notion he called the Archimedes integral for the integral of a function taking values in an infinite-dimensional space. This appeared in the proceedings, dedicated to Igor, of the conference held in September 1986 at Macquarie University. At this conference Alan McIntosh spoke about Igor’s work, and said something along the following lines:

Although Igor Kluváněk has recently resigned from his position at Flinders, this does not seem to have stopped his enthusiasm for research . . . his recent work might be described as developing an integration theory to cope with difficult and demanding situations not covered by earlier approaches.

The “Archimedes” approach to difficult integration situations was used by Igor in [26, 43, 46], where he widened the scope of integration theory with the idea of bringing a unifying approach to the theories of integration and summability, which he regarded, in their existing state, as being “a situation of truly baroque complexity”. He also had in mind the work of G. H. Hardy, for he says in [46, page 409]:

At any rate, the integration theory analogue of Hardy’s book *Divergent Series* is yet to be written.

Apart from Susumu Okada, this approach influenced others, and in the 1996 book by Brian Jefferies [17] there is a tribute to Igor which also indicates this influence, and the fact that Igor worked with an over-arching philosophy:

This book is an outgrowth of ideas originating from I. Kluváněk. Unfortunately, Professor Kluváněk did not live to contribute to the project of writing up in a systematic form, the circle of ideas to which the present work is devoted. It is more than likely that with his input, the approach and areas of emphasis of the resulting exposition would have been quite different from what we have here. Nevertheless, the stamp of Kluváněk’s thought and philosophy (but not necessarily his *approval*), abound throughout this book.

Chapter 7 in [17], specifically shows the influence of the “Archimedes approach” to integration in more abstract and general contexts, as mentioned by Igor in [35, 36, 43, 44].

3.6 Further mathematical work and influence

As previously indicated, Igor also had a major influence on the development of Australian mathematics through his students and their subsequent work. His students were A. Geue, B. Jefferies, G. Knowles, S. McKee, C. Meaney, S. Morris, R. Nilsen, W. Ricker, S. Okada, M. Sears and A. K. Whitford. Some of their work already has been mentioned. Tony Whitford [65] published work on the vector measures form of the Hamburger moment problem, and on the Laplace transform in the vector measures context [66]. Steve McKee [52, 53] carried out related work on orthogonal expansions and transforms of functions in a vector context. This work was in the spirit of Igor’s earlier work in [22, 23]. Andy

Geue [8] did work on continuous linear operators. Igor's friend and colleague at Flinders, P. G. Dodds [6], carried out work concerning Igor's results on the range of vector measures and how they could be seen from the point of view of order properties in Banach spaces. The two of them also had intensive discussion and exchanges of ideas on the role of order and vector lattice properties as they applied to parts of [28] and beyond. Also, mention should be made of the books [18, 58], authored in the former case by Brian Jefferies, and in the latter case by Susumu Okada, Werner Ricker and Enrique Sánchez Pérez, where particular use is made of vector measures. But not all of these students were in Igor's areas of prime research interest. Sid Morris completed his PhD in record time and went on to become an authority in topological groups. Chris Meaney and Rod Nilsen went on to work mainly in harmonic analysis, and the latter received substantial comment from Igor concerning the work in [55]. Michael Sears went on to work in dynamical systems and ergodic theory and more recently has become also a writer of detective stories. Michael Cwikel, a friend of Igor's who was also a student at Flinders, went on to work in mathematical and functional analysis.

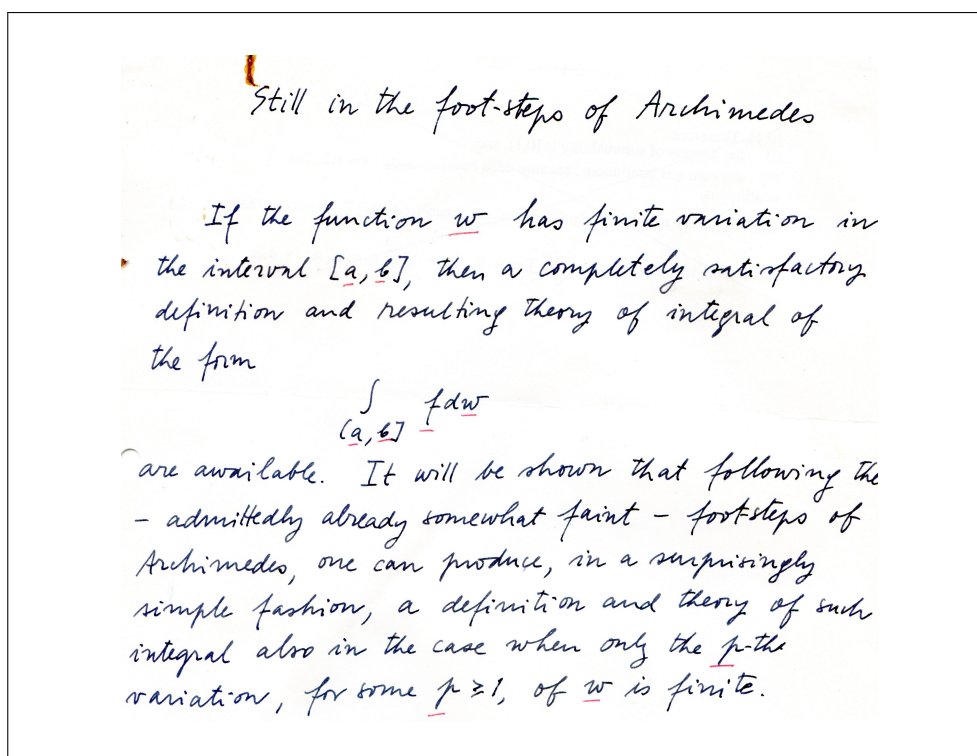


Figure 10. This is apparently an abstract of a paper Igor seems to have been working on in the mid to late 1980s.

4 Mathematical exposition and pedagogy

Igor Kluvánek maintained a steadfast interest and commitment to pedagogy, not only in mathematics, but in educational and pedagogical issues in general. In meeting him after a period, often the first thing he would mention would be a new way of looking at some mathematical result, often familiar and well known. That was what happened to me

in 1974, when I met him briefly in Trieste and he talked about a new proof of the inverse function theorem. However, he does not seem to have written on matters of teaching or exposition until the 1980s.



Figure 11. On a visit to Wollongong in the 1980s. From left to right: Rod Nilsen, Susumu Okada, Igor Kluváněk, Sid Morris, Michael Cwikel.

It was then that he took issue with the philosophy behind, and the content of, the matriculation examinations⁸ in South Australia [32]. The tenor of Igor’s criticisms was the lack of precision in how some questions were formulated, the intent behind the questions, and whether they developed independent skills or mere rote learning. He was also concerned at the notation used in the questions, which he considered in some cases to be unnecessarily complicated. Underlying these issues was the more general question of the content and spirit of the syllabus; not strictly a matter for the examination committee. I’m not aware that Igor received a detailed reply to his critique, but he was aware of attitudes along the lines that any discussion on the examinations could undermine confidence in them and that therefore discussion should not occur. The type of logic behind this reply was always anathema to Igor and his response was typical of his approach — that was, to concentrate on the underlying issue, not on the surface effects or consequences, nor on how a matter was commonly perceived:

... various measures proposed for a remedy are not always directed at the core of the problem
 ... attempts to improve the situation ... are sometimes stifled by the view ... that a criticism
 of the current procedures ... would undermine the confidence of students and, hence, should
 not be allowed. I will spare you my comments about the moral admissibility of such a position.

Igor’s papers [35, 36] on Archimedes’ approaches to calculating areas have been mentioned above. Igor’s paper “What is wrong with calculus?” [41] also has an historical dimension to it. He wrote:

⁸These were the examinations held in the final year of high school study, determining entry into university. I’m not aware that the term is any more used in the education systems in Australia.

The common view has it that, whatever there was to be done in this respect, was already taken care of by people like Cauchy, Bolzano, Weierstrass and their contemporaries. While there is a substantial element of truth in this view, it is not the whole truth . . . I will try and indicate some themes that still deserve our attention. In particular, I wish to suggest that the rôle of the notion of limit in elementary teaching can and should be substantially reduced.

Again, Igor is looking to the past and the historical development of the subject to give new directions. And again, he was not thinking of the past as a static museum exhibit. He draws attention to the difficulties in G. H. Hardy's proof [11] of the chain rule, a fundamental result in calculus. He writes:

Since its invention, the foundations of the differential and integral calculus were clarified and also the techniques were improved. And that gives us a key for the understanding of the deficiencies in the teaching of 'calculus'. The processes of clarification, simplification and improvement stopped at a certain stage and/or were confined to some aspects of the differential and integral calculus.

So having expressed the view that the ideas in integration theory had followed a particular idea of Archimedes, and that development of those ideas had stopped at a certain point, he also argued that a similar thing happened with the development of the calculus, at least at the pedagogical level.

Now, a function describes the change in a phenomenon over time, say. The function is continuous when small changes in time lead to correspondingly small changes in the function values. The calculus is concerned with the *rate* at which a function changes. The rate of change may vary from instant to instant, and to deal with such problems is one of the main reasons for the calculus of Newton and Leibniz. And yet, to estimate the rate of change of a function *at* an instant requires us to consider the values of the function *near* that instant. Indeed, there is a need to consider values of the function at times getting closer and closer to the given instant. This leads us to the notion of a *limit*, which nearly always appears in the definition of the derivative of a function and, indeed in the definition of a continuous function. In [41], Igor proposes that continuity of a function, at the elementary level on the real line, be defined in the same way as it is in topological spaces. This avoids the introduction of limits in discussing the notion of continuity, while retaining an intuitive grasp of the meaning of continuity. Then, the differentiability of the function is to be defined in terms of the existence of a certain continuous function. His definition is very simple. If f is a real valued function and x is a real number, f is *differentiable at x* means that there is a function φ , continuous at 0, such that for all u in some interval about 0,

$$f(x + u) - f(x) = \varphi(u)u.$$

Then, $\varphi(0)$ is the derivative of f at x . This definition of the derivative of f at x avoids the notion of limits.

The other main area of calculus where limits appear is in the definition of infinite sums, and calculating them. Here, Igor takes up the approach in his papers on Archimedes [35, 36]. That is, as mentioned before, in defining the infinite sum of a sequence (u_n) of non-negative numbers he says that the sum of the sequence is the smallest number s , if it exists, such that

$$u_1 + u_2 + \cdots + u_n \leq s, \text{ for all } n = 1, 2, 3, \dots$$

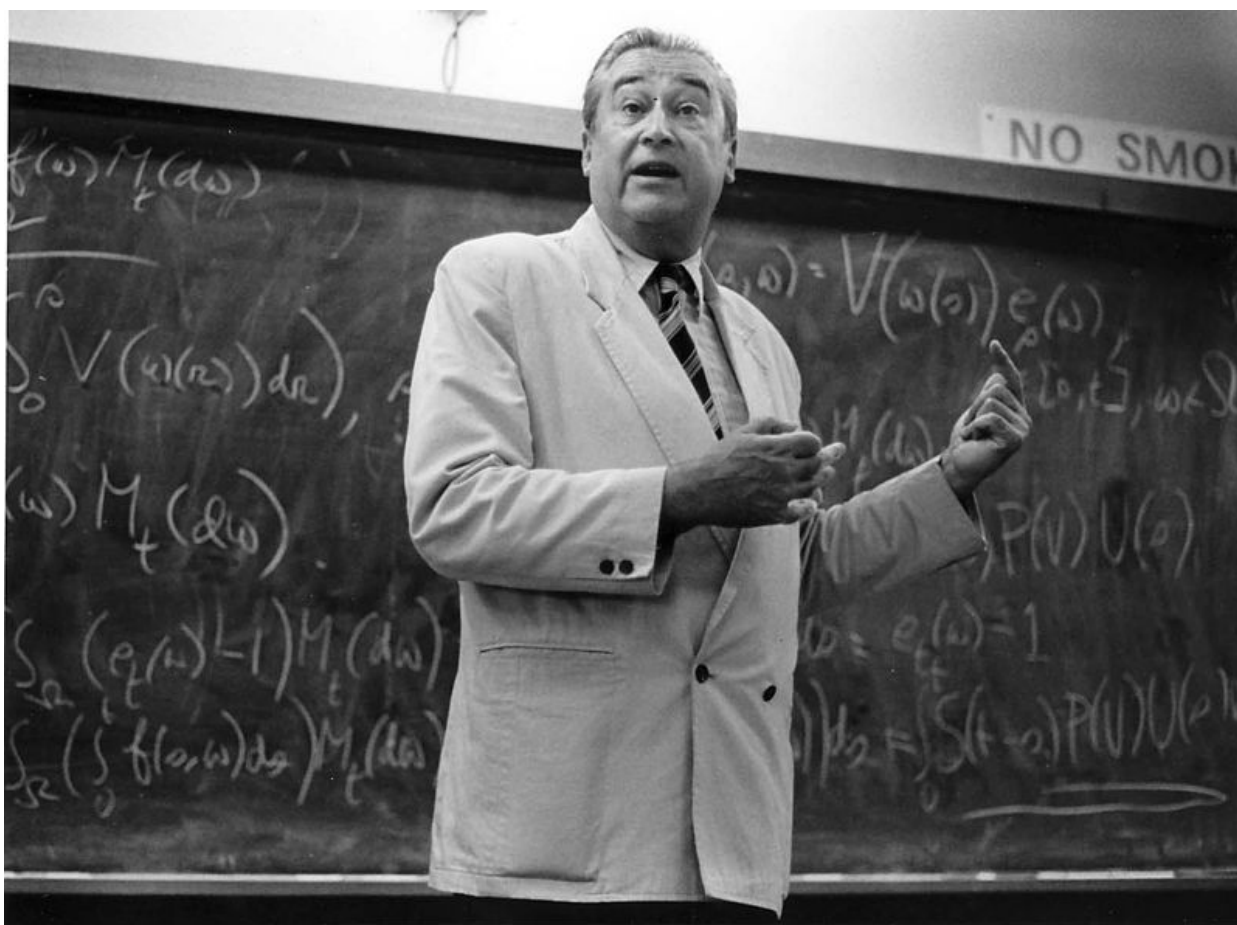


Figure 12. Igor Kluvánek in full flight. The photograph was taken in 1981 at the University of Queensland, and is included with the permission of Brian Jefferies.

Again, this definition obviates the need for the concept of an infinite series, and of the need for the concept of limits.

In suggesting these different pedagogical approaches to calculus, Igor retained a strong sense of the practicalities and did not go to an extreme. While expressing concern "... that certain inconsistencies may present serious, even unsurmountable, obstacles to those who try to understand that discipline ...", he also says "... an obsession with strict consistency of terminology and notation of a learned discipline is not a desirable attitude." Thus, he is in favour of retaining the notion of a variable, and of retaining the phrase "a function of the real variable", but he is concerned to say "... it is ridiculous to use the formal terms of set theory [which give clarity to the notion of a function] and, simultaneously, to leave the notion of a variable vaguely unexplained or even to define a function in its terms."

I'm not sure precisely to what extent these ideas of Igor's have been used in teaching calculus at the early years of university study. A contributing factor may be that his paper [41] was not published beyond being a research report at the Australian National University, as far as I am aware. I don't think there are any valid mathematical or pedagogical reasons

for not adopting Igor's approach to teaching calculus but, in Australia at least, the practical difficulties can be very considerable. This is because calculus classes may be large, in which case they involve the co-operation of many people, and there is often a resistance to becoming familiar with new ideas and putting in the time to implement them. Igor himself was fully aware of this type of situation. However, Igor's approach to infinite sums was used in [56], where Igor's ideas on the issue are discussed. Also, there has been some use of the 'Archimedes-Klůvák' approach to integration, for Brian Jefferies has used it in a later-year course at the University of New South Wales and Susumu Okada has used it in an honours course on integration at the University of Tasmania. Also, I understand that this approach to teaching integration in some first and second year courses has been used at Flinders University and the University of New South Wales respectively by Peter Dodds and Tony Dooley.

The papers on Archimedes and on calculus [35, 36, 41] came after an extended period Igor had at Flinders University on reflection and writing notes on calculus. In particular, Igor had frequent casual discussions with Werner Ricker about the calculus notes in the 1980s. A substantial part of these notes has now appeared in Slovakia as the two volumes *Calculus I* and *Calculus II* [48, 49]. Also, recent works dealing with or mentioning Igor's ideas on calculus are [9, 10, 63].

Apart from mathematics, Igor had views on general pedagogy. In his paper "The idea of cognition in Christian school" [45], he identified what he considered to be the two main errors currently dominant in the perception of the nature of human cognition. These were what he called "the machine model" and the "divine model". His view was that these errors derived from true aspects of human cognition, but were flawed because they were isolated from the whole picture, and amplified to the point of caricature. Concerning the pedagogical aspects of these notions he said [45]:

From a perspective dominated by the machine model of cognition, the task of a teacher is to provide information and the task of a learner (pupil) is merely to note and retain it ... the divine model of human cognition is the Cartesian dictum 'Cogito Ergo Sum' driven to its extreme, often absurd, consequences ... whenever I accept anybody's opinion, be it a single person, possibly myself, or a group of persons, even a large group, without understanding how this opinion is related to some human experience, I am influenced by this model.

Although he considered these two views had been rejected within philosophy itself, he thought they were still dominant among educated professionals across the various fields. These considerations posed the question: if each of these views is incomplete and isolated from the whole picture, what is this whole picture and how is this general question of a valid epistemology and a proper model of cognition to be resolved?

Here, Igor draws on the work of Stanley Jaki who, in his Gifford Lectures of 1974-75 and 1975-76 [14], argued that the rise of modern science in Europe was due, in no small part, to the Pauline works of the Epistles to the Romans and the Galatians, in particular. Igor goes on to argue further in [45] that the Johannine Gospel should be equally stressed in this regard. He considers that various passages in this Gospel show that its main concern is the problem of knowledge, and he argues this at length with specific and detailed references to the text. This exposition is closely related to his earlier essay [38] on human cognition and St John's Gospel.

5 Philosophical and religious thought

For mathematicians, it is easy to restrict our views and thoughts of Igor to his mathematical side. After all, to mathematicians, what could be more important about a mathematician than his or her mathematical achievements and ideas? But in Igor's case, even some of his mathematical writings are scattered with references to philosophical ideas, and some of these are quite developed, especially considering the research context in which they occur. He did not start to write on philosophical and religious matters, it seems, until the 1980s; but it is clear that the thoughts he then expressed had been in gestation for a long time. Also, Igor was inclined naturally to philosophical analysis and to analysing society and the intellectual environment, wherever he was.

A frequent characteristic of discussion on policy matters in Australia is that there may be little distinction drawn between a policy or a set of proposed actions, and the incidental effects that the policy or actions may have. So, discussion about short-term side effects frequently obscures discussion of substantial underlying issues. It was a characteristic of Igor's that invariably he put what he considered to be the fundamental, substantial issues at the front of discourse; incidental and peripheral effects were never allowed to obscure what he considered the main discussion should be about. Nearly always his inclination was to examine the philosophical assumptions underpinning a situation, and to consider the consequences. This gave his thought and discussion of issues a highly intellectual quality.

In about 1986, some prominent Australian Catholics were invited to participate in the Australian contribution to the 1988 Assemblée Plénière, Secretariat for Non-Believers. Igor was one of these. The participants were to respond to five questions, and Igor responded to at least two [37]. These two questions were: (i) "What are the dominant ideologies and mentalities in Australia? What is their impact on the thought and life-style of the Australian of today?" (ii) "What pastoral suggestions would you offer for presenting the faith and the essentials of Christian living in a world affected by today's various ideologies and mentalities?" Consistent with Igor's general outlook, his response to these questions goes to fundamentals, especially the underlying epistemology and the perceived nature of cognition. His critical attitude to many things happening in Australia was unmistakable. He considered positivism to be the philosophical and epistemological enemy, and considered that the positivistic outlook should be combatted more effectively. What did Igor mean by positivism? He expressed his view succinctly in [37]:

Every form of positivism elevates some facts to the dignity of principles, or axioms, to which all arguments in a given sphere of discussion are reduced. Such facts are thereby removed from among the targets of human cognitional activity. So, for example, if such a fact happens to be an institution, then this institution is made immune against intelligent understanding and criticism. Thus, every form of positivism is a reactionary force; it is both a cause and manifestation of deeply seated conservatism.

He also considered that positivism produced an undue scepticism, and cited the case of the physicist Ernst Mach (1838-1916), who opposed the atomic theory on the basis of his positivistic philosophy. In Igor's view, positivistic attitudes were unwittingly taken up, without sufficient awareness of what they represented, and which took little account of the history of discussion of related issues which were within the intellectual traditions of the

Church and wider society. For example, in responding to question (ii) he wrote (and note Igor's combative mode of expression at one part, and his application of these comments to Australian society as a whole):

... there are often grounds for a suspicion that what is being promoted ... as new is just a ... set of vague attitudes without any articulated position and which would not withstand even the most superficial scrutiny ... one can often hear pronouncements ..., but nobody making them is bothered to explain what is (or was?) the point of transcendental Thomism, say. Such people do not even give signs of knowing that it may have any bearing on the pronouncements they are making, let alone whether the programme of transcendental Thomism is viable or proved to be a dud. It is not necessary to go into further details because this phenomenon is well-known and has analogues in all sections of the cultural life of this Country.

This is indicative of an uncompromising directness which is evident in some of Igor's writings, and in his character. It also indicated his view that problems for the Church in Australia derived from the corresponding problems in society and that, to avoid this, a more independent and critical attitude should be taken to the norms of discourse and the prevailing attitudes and values in society. In this, Igor was quite critical of certain Roman Catholic environments in which he found himself in Australia. But I think his attitude to this situation is described by the remark of the English Catholic writer G. K. Chesterton when he wrote in [2, page 113]:

The devotee is entirely free to criticise; the fanatic can safely be a sceptic.

I suspect that Igor's critiques were largely received with incomprehension, and consequently ignored. In fact, the indifference and incomprehension with which his submission apparently was received was one of the very few occasions when I saw Igor reveal a distinct disappointment. Mention should also be made of Igor's other writings [34, 38, 39, 45] which, to differing extents, are philosophical and religious.

It is one thing to identify the problems of positivism, and to oppose a positivistic outlook, but it is another to have a very specific religious commitment, of the type which Igor exemplified. Following Igor's death, Brian Jefferies mentioned the deep, personal loss many of those who knew him felt, and wrote [16]:

... there is a great cultural loss—with a serene old age, he may have had time to systematically write down his original views on philosophy and education.

As it is, in so far as I know about them, Igor's philosophical and religious writings are fragmentary and not systematically developed in the way that his mathematics was, and there was the real possibility that he might have gone on to write extensively on these wider ideas. But even as it is, his thoughts on these matters are a substantial contribution to philosophical, social and religious analysis, especially pertaining to the situation in Australia. Here, perhaps note particularly [37, 45]. Certainly, his thoughts on, and analysis of, political, cultural and educational issues in Australia remain highly pertinent, and by 1987 he had identified the seeds in Australia which produced the subsequent upheaval in higher education, commencing with the Labor Government's policies under John Dawkins, the Commonwealth Minister for Education, in 1988.

6 Attitudes and personality

The European intellectual is a rarity in Australia, and Igor's commitment to ideas and culture, both high and sometimes popular, across a wide range of serious interests did not fit into any recognised categories. Igor always took ideas very seriously, believing that even abstract false ideas produced bad practical consequences. That seems to have been why he placed so much importance on the concept of cognition at work in a situation, and why his intellectual and social analysis always went to the underlying issues. This intellectualism was the main reason, I think, why his non-mathematical writings tended to be received with incomprehension, and why Igor's acceptance by the intellectual environment in which he found himself in Australia was inevitably problematic. As the late Eugene Kamenka of the Australian National University said, in writing about Australia [19]:

Their public philosophy was narrow, insular, and anti-intellectual. It required not so much belief in dogma and superstition as an Anglican reluctance to follow principles to the extent of ... rocking the boat. To stand out was to invite suspicion; to treat ideas seriously was to waste one's time, to display a fundamental sense of unreality.

In Australia, it was not uncommon for Igor to be at loggerheads with what he considered to be the positivism, pragmatism and spirit of compromise that he thought to be very much a part of the Anglo-Saxon tradition. Indeed, he thought that this was part of Australia's inheritance as a former group of British colonies. Although it was not characteristic of Igor to set out deliberately to "rock the boat", if he felt that an argument or a conclusion produced a clear position or course of action, then he would argue for it with vigour.

In intellectual matters, Igor did not allow his feelings to affect his judgment. I think the not uncommon perception that Igor was uncompromising in his views stemmed in large part from his preparedness to distinguish between judgment and feeling in general, and from his placing more importance on correct judgment than on his own feelings in particular. He tended not to take account of the "politics" of a situation, and could give the impression that, in his view, no serious person should do so. In referring to his time at St Dominic's in Melbourne, saying that he had enjoyed his time there, he also said [40]:

...because this is a matter of purely subjective attitude, there is no point to dwell on it any further.

This is the only place where I am aware of his expressing explicitly the distinction between feeling and judgment as it applied to himself, but it shows his very strong sense of adherence to principle, realism and a commitment to putting truth ahead of his own personal feelings. Igor's adherence to principle has been already noted by B. Riečan in [62]. Igor had definite views on many things, and he could express those views in a very definite manner. There were people who sometimes regarded him as overbearing. He was often controversial and not necessarily popular. But he could also show great support for and loyalty to people. In ordinary conversation, Igor had a great enthusiasm for ideas, discussion and debate, which he often conveyed with a characteristic emphasis, frequently combined with humour. He achieved moments of great enthusiasm and intensity in his lectures and seminars, and he conveyed the feeling that the ideas were really significant, not because they came from him, but because of their intrinsic value and qualities. But also, as noted by Brian Jefferies [16],

Igor could be enigmatic, sometimes only alluding to his opinion or an argument, with the apparent expectation that it was up to the person to think it out for themselves. Socially, Igor could be charming and entertaining, and show an interest in people's lives and concerns. He was a complex personality whose adherence to principles and to the seriousness of ideas made him an inspiration to many who knew him.

Igor's characteristic sense of humour frequently arose from a sudden switch between the "high" and the "low". Here are some examples from [30, 37, 41]:

... whether the programme of transcendental Thomism is viable or proved to be a dud.

He [Professor McShane] gives me an opportunity to put in my usual commercial slogan: Vector measures to every household,

I would like to finish these general remarks with one which is almost serious.

I will skip the commercial about the further possibilities in this approach to the derivative.

In the Australian political context, Igor was generally of the "left", but not in a conventional sense. Politically, he was in that part of the Catholic tradition that emphasises social concern and the dignity of humanity which, in Australia, has been represented historically at the political level by the labor movement. Igor's rigorous view of knowledge, however, and his religious views and orthodoxy, also meant that his outlook had a conservative aspect. He was often critical of simplistic labels or intellectual positions, such as the common dichotomy often drawn between the 'conservatives' and the 'progressives', an issue discussed in [39]. As noted by Brian Jefferies [16], Igor admired the writing of G. K. Chesterton, and frequently quoted him. Chesterton's use of paradox and sense of irreverence appealed to Igor, so it is appropriate that it is Chesterton [3] who, it seems to me, has expressed something of Igor's attitude to conventional political and intellectual categories:

The whole modern world has divided itself into Conservatives and Progressives. The business of Progressives is to go on making mistakes. The business of the Conservatives is to prevent the mistakes from being corrected.

Igor's complexity of outlook, which applied to his scientific work, his political and religious views, and to his views of society, meant he did not fit easily into the conventional categories of discourse, especially as it is not usual for mathematicians in Australia to have highly developed views on such a wide range of issues. In his concern to integrate ideas and concepts across human experience, he was in opposition, as a person and as a thinker, to the human and intellectual fragmentation which is such a feature of post-modern Western society and, it is clear, he considered that to be the case for Australian society in particular.

Despite Igor's definite and sometimes uncompromising views, it was an error to assume that one was expected to accept his views simply because they were his opinion – rather, as one got to know him, he gave the deeper impression that he was expressing something beyond his own opinion, and that each person should consider carefully to ascertain the truth. The enigmatic impression he sometimes gave for many of those who knew him in Australia was partly a result of what could be termed this "transcendent" view of knowledge, and this was not necessarily tempered even when he chose to express himself in "plain Slovak", as he frequently did.

7 Conclusion

All political philosophies contain their own deficiencies and their own pitfalls. And all intellectual disciplines have their own characteristics and encourage their own modes of thought, which are also deficient and have their own pitfalls. In mathematics, one might say that the most obvious pitfall stems from the extreme technical demands the discipline makes upon its practitioners. It may degenerate into a series of technicalities which, important and crucial though they must be, can sometimes obscure mathematics as an activity of free and open enquiry, despite being one that does not abide personal preferences or wishful thinking.

Igor Kluvánek was a rare type of individual who did not allow the technical demands of his mathematical work to enclose him. But Igor's engagement with the wider picture did not lead to confusion, whereby little or nothing can be resolved. Rather, he followed his own clear course, always independently, balancing an acute reasoning faculty with a commitment to intellectual and religious faith, a combination which was quite rare in Australian intellectual life. Concerning the sciences he wrote [30, page 103]:

The possibility of applications of results of one branch of science in another one, or of the results of a science in another science, presupposes an internal unity or interdependence among various sciences. I believe . . . that such a unity and interdependence really exists.

Concerning understanding he said:

And yet, a person understands some information available to him or her only if he or she grasps the connections, the relationships, between phenomena, concepts and ideas to which the information refers. It can be said that the understanding of information consists precisely in the grasping of such relations.

Igor carried his belief in an internal unity and interdependence between the sciences more broadly into life as whole. That is why one does not feel a marked divide between his mathematical research and his later pedagogical, philosophical and religious writings. His intellectual outlook remained calm and unified across what is often regarded as a chasm, between knowledge and faith.

There are two thoughts which I find apposite in thinking about Igor and his life. One is from a writer Igor admired and recommended, Henri de Lubac. In his *Paradoxes of Faith* [4, page 112], de Lubac writes:

It is by dint of impersonality that . . . [serious thought] . . . makes a conquest of itself and, without seeking to do so, becomes personal.

Also, the Australian philosopher Raymond Gaita says [7, page 40]:

There is no way, however, to the understanding of such value [of a philosophical and enquiring life] than one mediated by authoritative example, for there is no purely discursive route, available to any reasonably intelligent person of goodwill, to the appreciation of the value of anything important, let alone the value of a life devoted to the pursuit of understanding.

In times where the serious life of the mind struggles to exist in the corporate institutions which universities in the West have become, Igor's life and example, with their dedication to principle and truth in their broadest senses, involving the whole person, remain inspiring.

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