

School of Mathematics & Applied Statistics
**MATH111: Mathematics Applied Mathematical
 Modelling 1**
Assignment Week 4 Solutions
Spring 2007

1. At the beginning of each year you invest $q\%$ of your yearly salary into a superannuation scheme. Your employer tops up your investment by adding $r\%$ of your salary. Your money draws interest of $p\%$ compounded yearly.
 - (a) Write down a **word** equation that defines your investment after n payments.
 - (b) Suppose that your salary starts at a base level a and that at the start of every year it increases by a constant amount b .

Solution

(a)

$$\left\{ \begin{array}{l} \text{Money after} \\ nth \text{ payment} \end{array} \right\} - \left\{ \begin{array}{l} \text{Money after} \\ (n-1)\text{th payment} \end{array} \right\} = \left\{ \begin{array}{l} \text{Interest earned on} \\ \text{investments after } (n-1) \text{ years} \end{array} \right\} + \left\{ \begin{array}{l} \text{Investment at the start} \\ \text{of year } n \end{array} \right\}.$$

- (b) Write down, formally, the difference equation that describes your investment after n payments have been made. Define **all** variables and explain your terms.

Your salary at the beginning of year n is $a + nb$.

Your investment into your superannuation scheme at the start of year n is $\frac{q+r}{100} \cdot (1 + nb)$

Let money after n payments be M_n .

Let annual compound interest be p .

Then

$$\begin{aligned} M_n - M_{n-1} &= \frac{p}{100} M_{n-1} + \frac{q+r}{100} (1 + nb), \\ \Rightarrow M_n &= \left(1 + \frac{p}{100}\right) M_{n-1} + \frac{q+r}{100} (1 + nb). \end{aligned}$$

2. A debt of \$50,000 with interest at 11% compounded every two months is paid back by equal payments every two months over fifteen years.
 - (a) What is the regular payment, rounded up to the nearest cent?
 - (b) How much profit does the lender make on the loan?

Solution

- (a) The loan repayment formula is given by

$$\begin{aligned} D_n &= (1+i)^n \left(D_0 - \frac{R}{i} \right) + \frac{R}{i}, \\ i &= \frac{\alpha p}{100}. \end{aligned}$$

When the loan is repaid we have $D_n = 0$. The loan repayment formula can be rewritten to give

$$R = \frac{i(1+i)^n D_0}{(1+i)^n - 1}.$$

We have

$$\begin{aligned} D_0 &= 50,000 \\ p &= 11, \\ \alpha &= \frac{1}{6}, \\ i &= \frac{\alpha p}{100} = \frac{\frac{1}{6} \times 11}{100}, \\ n &= 6 \times 15 = 90. \end{aligned}$$

Putting these numbers into the formula for R we obtain that

$$R = \$1138.64.$$

(Note. It is *wrong* to round the answer down to \$1138.63.)

(b) Profit = $90 \times 1138.64 - 50000 = \$52476.99$.

3. “If the Indians hand’t spent the \$24! In 1626 Peter Minuit, the first governor of New Netherland, purchased Manhattan Island from the Indians for about \$24.” Quote 2130 in [1].

Assume that the Indians had invested their \$24 in 1626 and that interest is compounded annually at a uniform rate of 7%. (For the 280 years to 1906, this percentage is about average for the stockmarket when it is not crashing [1]).

[1] R.I. Moritz. Memorabilia Mathematica: the Philomath’s Quotation Book. Mathematical Association of America. (1993)

- (a) What would the value of the Indians investment be in 1906?
 (b) What would the value be at the end of today? (Make the same assumption regarding the returns over the investment period).

Solution

- (a) We use the compound interest formula

$$\begin{aligned} S_n &= \left(1 + \frac{\alpha p}{100}\right)^n S_0, \\ S_0 &= 24, \\ \alpha &= 1, \\ p &= 7, \\ n &= 280, \\ \Rightarrow S_n &= \$0.4052 \times 10^{10}. \end{aligned}$$

That’s a lot of money!

- (b) We use the compound interest formula

$$\begin{aligned} S_n &= \left(1 + \frac{\alpha p}{100}\right)^n S_0, \\ S_0 &= 24, \\ \alpha &= 1, \\ p &= 7, \\ n &= 381, \\ \Rightarrow S_n &= \$0.3762 \times 10^{13}. \end{aligned}$$

That’s even more money! (You might decide to take $n = 380$ in this calculation).

4. “Properties that were out of reach with a 30-year mortgage can suddenly become affordable when the repayments are spread over an extra 10 years”. The Sydney Morning Herald (June 21, 2006)

- (a) A loan of \$250,000 is taken to purchase a house. Suppose that interest is compounded monthly at 7%.
- (i) What is the monthly repayment if the loan is paid back over a 30-year mortgage?
 - (ii) What is the monthly repayment if the loan is paid back over a 40-year mortgage?
 - (iii) How much more does it cost to repay the loan over 40 years?
- (b) Suppose that you have a mortgage at which interest is compounded monthly at 7% and that you can afford to pay \$1700 a month.
- (a) If you wish to pay the loan back over 30 years how much money can you borrow from the bank?
 - (b) If you wish to pay the loan back over 40 years how much money can you borrow from the bank?
 - (c) How much more money can you borrow from the bank if you spread your repayments over forty years?
 - (d) By taking a 40-year mortgage how much more money do you end up paying the bank?
 - (e) Do you agree with quotation from The Sydney Morning Herald?

Solution

- (a) (i) The loan repayment formula is given by

$$D_n = (1+i)^n \left(D_0 - \frac{R}{i} \right) + \frac{R}{i},$$

$$i = \frac{\alpha p}{100}.$$

When the loan is repaid we have $D_n = 0$. The loan repayment formula can be rewritten to give

$$R = \frac{i(1+i)^n D_0}{(1+i)^n - 1}.$$

We have

$$\begin{aligned} D_0 &= 250,000 \\ p &= 7, \\ \alpha &= \frac{1}{12}, \\ i &= \frac{\alpha p}{100} = \frac{\frac{1}{12} \times 7}{100}, \\ n &= 12 \times 30 = 360. \end{aligned}$$

Putting these numbers into the formula for R we obtain that

$$R = \$1663.26.$$

(Note that we must round the payment up).

- (ii) Repeating the previous calculation but using $n = 480$ we obtain $R = \$1553.58$.
 - (iii) $480 \times 1553.58 - 360 \times 1663.26 = \$146944.80$
- (b) (i) Use the formulae

$$D_n = (1+i)^n \left(D_0 - \frac{R}{i} \right) + \frac{R}{i},$$

$$i = \frac{\alpha p}{100},$$

$$D_n = 0.$$

Rearrange the formula to obtain

$$D_0 = \frac{R}{i} \left[1 - \frac{1}{(1+i)^n} \right].$$

We have

$$\begin{aligned} R &= 1700, \\ p &= 7, \\ \alpha &= \frac{1}{12}, \\ i &= \frac{\alpha p}{100} = \frac{\frac{1}{12} \times 7}{100}, \\ n &= 12 \times 30 = 360. \end{aligned}$$

Putting these numbers into the formula for D_0 we obtain that

$$D_0 = \$255522.86 \approx \$255523.$$

- (ii) Repeating the calculation with $n = 12 \times 40 = 480$ we obtain $D_0 = \$273562.0233 \approx \$273562$
 - (iii) You can borrow an additional $273562 - 255523 = \$18039$.
 - (iv) You pay back the bank an extra $1700 \times 12 \times 10 = \$204000$.
 - (v) There isn't a 'correct' answer to this question. But whether you said 'yes' or 'no', you need an a good reason to support your position.
5. In the mid-session test and/or the final exam you may be asked a question about Maple.

This question involves finding the wavelength, L , of a tsunami for a given water depth [2]. The governing equation is

$$f(L) = L - \frac{gT^2}{2\pi} \tanh\left(\frac{2\pi d}{L}\right) = 0, \quad (1)$$

where g is gravity, $T = 2880$ s is the period of the tsunami, and $d \approx 4000$ m is the water depth typical for the Indian Ocean [2].

- (a) Plot the function $f(L)$ and graphically estimate the zero. Make sure that your plot contains appropriate axis labels.
Note. In plotting the function you will need to decide upon a suitable horizontal range.
- (b) Use `fsolve` to find the zero.
Note. You may need to give maple some 'help' to find the zero. Look at the help page for the `fsolve` command.
- (c) (i) Explain the approximation $\tanh x \approx x$.
(ii) Hence, or otherwise, obtain the approximation

$$f(L) \approx L - \frac{gT^2 d}{L}.$$

- (iii) Solve this equation for L_1 .
- (iv) Compare the approximate answer L_1 against the full numerical solution. What is the error?
Note. Do not use maple for any part of this question.
- (d) (i) Explain the approximation $\tanh x \approx x - \frac{x^3}{3}$. (Do not use maple for this question)
(ii) Hence, or otherwise, obtain the approximation

$$L^4 - (gT^2 d) L^2 + 4gT^2 d^3 \pi^2 \approx 0$$

and use the `solve` command to find the zeroes of this function. Which of these zeroes is the correct answer to the physical problem?

- (e) If $d = 4000 \pm 50$ what error does one get using the linear approximation L_1 ?

[2] S.I. Barry & T. Webb. Multi-disciplinary approach to teaching numerical methods to engineers using Matlab. *ANZIAM Journal* **47**, C216–C230, 2006.

Solution The maple code used in this question is given at the end.

- (a) In order to draw an appropriate graph you need to experiment with the horizontal range of the function. The point of this question is sometimes you want to plot a function, but you do not what range to use — you need to do some preliminary investigation. The required figure is shown in figure 1.

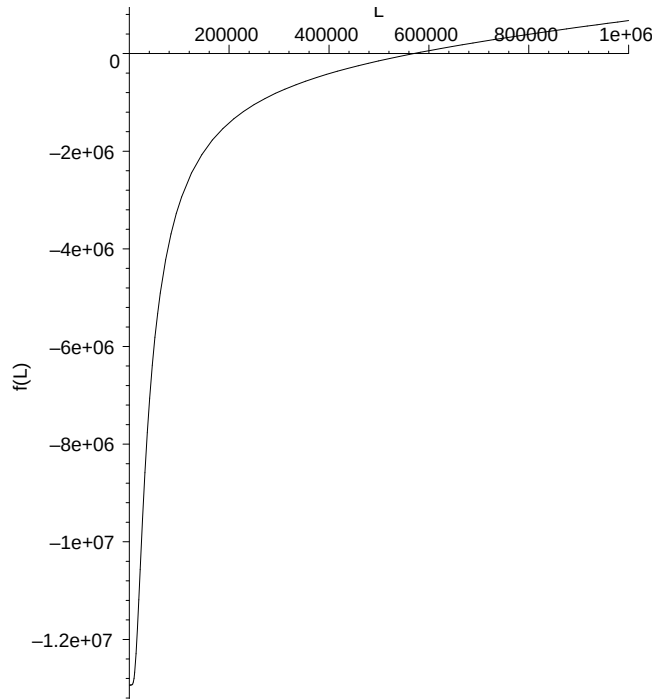


Figure 1: Figure showing $f(L)$ as a function of the wavelength L .

- (b) Giving maple a little help — based on what we can estimate from the figure — we obtain that $L = 570026$.
- (c) (i) Use a Taylor series expansion around the point $a = 0$.

$$\begin{aligned} f(x) &\approx f(0) + f'(0)x + O(x^2), \\ f(x) &= \tanh x, & f(0) &= 0, \\ f'(x) &= \operatorname{sech}^2 x = 1 - \tanh^2 x, & f'(0) &= 1, \\ &\Rightarrow \tanh x \approx x. \end{aligned}$$

- (ii) Using the result that $\tanh x \approx x$ we have

$$\tanh\left(\frac{2\pi d}{L}\right) \approx \frac{2\pi d}{L}.$$

Thus

$$f(L) \approx L - \frac{gT^2}{2\pi} \cdot \frac{2\pi d}{L} = L - \frac{gT^2 d}{L}.$$

- (iii) From our answer to the previous part of this question we obtain the approximate solution

$$L \approx \sqrt{gT^2 d}.$$

- (iv) The approximate answer gives $L_1 \approx 570,210.9083$.
- (v) The percentage error in the approximate answer is only 0.03%!

- (d) (i) Use a Taylor series expansion around the point $a = 0$.

$$\begin{aligned}
 f(x) &\approx f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{2!}f'''(0)x^3, \\
 f(x) &= \tanh x, & f(0) &= 0. \\
 f'(x) &= \operatorname{sech}^2 x = 1 - \tanh^2 x, & f'(0) &= 1, \\
 f''(x) &= -2 \tanh(x) \operatorname{sech}^2(x) = -2 \tanh(x) (1 - \tanh^2(x)) & f''(0) &= 0, \\
 f'''(x) &= 4 \tanh^2(x) [1 - \tanh^2(x)] - 2 [1 - \tanh^2(x)]^2 & f'''(0) &= -2. \\
 \Rightarrow \tanh x &\approx x - \frac{1}{3}x^3 + O(x^4).
 \end{aligned}$$

- (e) Using the approximation that $\tanh x \approx x - \frac{x^3}{3}$ we have

$$\begin{aligned}
 f(L) &= L - \frac{gT^2}{2\pi} \tanh\left(\frac{2\pi d}{L}\right), \\
 &\approx L - \frac{gT^2}{2\pi} \left[\frac{2\pi d}{L} + \frac{1}{3} \left(\frac{2\pi d}{L}\right)^3 \right], \\
 \Rightarrow 0 &= L^4 - g d T^2 L^2 + \frac{4}{3} \pi^2 g T^2 d^3.
 \end{aligned}$$

- (f) The equation has four real roots. Two are negative and therefore not of interest. Of the two real roots, the one of interest is the one nearest our earlier answers. That is, $L = 570026.132$. This is an excellent approximation to the correct answer!
- (g) When $d = 3950$ the exact answer is given by $L = 566454.7$. When $d = 4050$ the exact answer is given by $L = 573575.7$. When $d = 4000$ the approximate answer was $L_1 \approx 570,210.9083$. The moral of this is that if all we know is that $d = 4000 \pm 50$ then for any practical purpose the approximate answer is good enough.

```

#week04-2007.maple
#
f := L - g*T^2/(2*Pi)*tanh(2*Pi*d/L);

g := 9.8;      # gravity
T := 2880;     # period of the tsunami
d := 4000;     # average water depth for the Indian ocean

plot(f,L=0..10^6,labels=["L","f(L)"],\
      labeldirections=[horizontal,vertical],thickness=1,color=black);

fsolve(f,L); # this command gives me a negative answer! The negative answer
              # can not be the correct physical answer. But I can see from
              # the graph that the answer is at approximately 600,000.
Lcorrect := fsolve(f,L=600000);

# calculate the approximate answer

Lapprox := sqrt(g*T^2*d);

# percentage error in the approximate answer

Lerror := 100*(Lapprox-Lcorrect)/Lcorrect;

# The improved approximate answer

```

```
Leqn := L^4 -g*T^2*d*L^2 +4*Pi^2*g*T^2*d^3/3;
```

```
solve(Leqn);
```

```
# Find the exact answer when d = 4050.
```

```
d := 4050;
```

```
fsolve(f,L=600000);
```

```
# Find the exact answer when d = 3950.
```

```
d := 3950;
```

```
fsolve(f,L=600000);
```