

VIII. TRANSPORT

1.8. Boltzmann Transport Equation

$$f(\vec{r}, \vec{v}) d\vec{r} d\vec{v} = \text{number of particles in } d\vec{r} d\vec{v}$$

In the absence of collision the distribution is conserved

$$f(t+dt, \vec{r}+d\vec{r}, \vec{v}+d\vec{v}) = f(t, \vec{r}, \vec{v})$$

With collision

$$f(t+dt, \vec{r}+d\vec{r}, \vec{v}+d\vec{v}) - f(t, \vec{r}, \vec{v}) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} dt$$

expand this term

$$f(t, \vec{r}, \vec{v}) + \frac{\partial f}{\partial t} dt + d\vec{r} \cdot \nabla_{\vec{r}} f + d\vec{v} \cdot \nabla_{\vec{v}} f - f(t, \vec{r}, \vec{v}) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} dt$$

or

$$\frac{\partial f}{\partial t} + \frac{d\vec{r}}{dt} \cdot \nabla_{\vec{r}} f + \frac{d\vec{v}}{dt} \cdot \nabla_{\vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$$

Boltzmann Equation.

Relaxation approximation

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{f-f_0}{\tau_c}$$

τ_c has the meaning of mean collision time

$$\text{If } \frac{\partial f_0}{\partial t} = 0 \Rightarrow \frac{\partial f}{\partial t} = \frac{\partial(f-f_0)}{\partial t} = -\frac{f-f_0}{\tau_c}$$

i.e. $(f-f_0) = f_1$

$$f_1 = f_1(t=0) e^{-\frac{t}{\tau_c}}$$

Back to diffusion:

$$\text{Steady state } \frac{\partial f}{\partial t} = 0$$

$$\text{if } \frac{\partial f}{\partial t} = 0, \quad \vec{v} \cdot \nabla f = -\frac{f-f_0}{\tau_c}$$

free space

$$\text{for one dimensional case } \vec{v} = v_x \quad \nabla f = \frac{\partial f}{\partial x}$$

$$v_x \frac{\partial f}{\partial x} = -\frac{f-f_0}{\tau_c}$$

$$\text{or } f = f_0 - \tau_c v_x \frac{\partial f}{\partial x}$$

$$\text{try iteration } f^{(0)} = f_0$$

$$f^{(1)} = f_0 - \tau_c v_x \frac{\partial f_0}{\partial x}$$

$$f^{(2)} = f_0 - v_x \tau_c \frac{\partial f_1}{\partial x} = f_0 - v_x \tau_c \frac{\partial f_0}{\partial x} + v_x^2 \tau_c^2 \frac{\partial^2 f_0}{\partial x^2}$$

Apply classical distribution

$$f_0 = e^{\frac{\mu - \epsilon}{\tau_c}}$$

$$\frac{\partial f_0}{\partial x} = \frac{\partial f_0}{\partial \mu} \frac{d\mu}{dx} = \frac{f_0}{\tau_c} \frac{d\mu}{dx}$$

$$\text{therefore } f = f_0 - v_x \tau_c f_0 \frac{d\mu}{dx}$$

particle flux density

$$J_n^x = \int v_x f D(\epsilon) d\epsilon, \quad D(\epsilon) = \frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon}$$

$$= \int v_x f_0 D(\epsilon) d\epsilon - \tau_c \frac{d\mu}{dx} \int v_x^2 f_0 D(\epsilon) d\epsilon$$

$\downarrow$
0
odd

$\downarrow$
nonzero

$$J_n^x = - \left(\frac{d\mu}{dx} \right) \frac{\tau_c}{\tau} \int v_x^2 f_0 \mathcal{D}(\epsilon) d\epsilon$$

$$= - \left(\frac{d\mu}{dx} \right) \frac{\tau_c}{\tau} \frac{1}{3} \int v^2 f_0 \mathcal{D}(\epsilon) d\epsilon$$

$$= - \left(\frac{d\mu}{dx} \right) \frac{\tau_c}{\tau} \frac{1}{3} \langle v^2 \rangle$$

$$= - \left(\frac{d\mu}{dx} \right) \frac{\tau_c}{\tau} \frac{1}{3} \frac{3n\tau}{M} = - \frac{n\tau_c}{M} \frac{d\mu}{dx}$$

remember $\mu = \tau \ln(n) + \text{const}$ $\frac{d\mu}{dx} = \tau \frac{1}{n} \frac{dn}{dx}$

$$J_n^x = - \frac{\tau_c \tau}{M} \frac{dn}{dx} = - D \frac{dn}{dx}$$

$$\text{if } \frac{1}{3} \langle v^2 \rangle \tau_c \gg \tau$$

D is the diffusion constant $D = \frac{\tau_c \tau}{M} = \frac{1}{3} \langle v^2 \rangle \tau_c$

Another assumption if τ_c is not simply a const. but

$$\tau_c = \frac{l}{v}$$

$$J_n^x = - \left(\frac{d\mu}{dx} \right) \left(\frac{l}{\tau} \right) \int \left(\frac{v_x^2}{v} \right) f_0 \mathcal{D}(\epsilon) d\epsilon$$

$$= - \left(\frac{d\mu}{dx} \right) \left(\frac{l}{\tau} \right) \frac{1}{3} \int v f_0 \mathcal{D}(\epsilon) d\epsilon$$

$\downarrow$
 n_c

$$c = \langle v \rangle$$

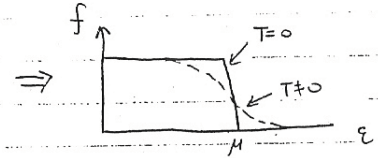
$$= - \frac{1}{3} n_c \left(\frac{l}{\tau} \right) \frac{d\mu}{dx}$$

$$= - \frac{1}{3} c l \frac{dn}{dx}$$

In this case $D = \frac{1}{3} c l$

Fermi-Dirac Distribution

$$f_0 = \frac{1}{e^{\frac{\epsilon - \mu}{\tau}} + 1}$$



At low temperature $\tau \ll \mu$

$$\frac{df_0}{d\mu} \approx \delta(\epsilon - \mu)$$

Any function $G(\epsilon) \quad \int G(\epsilon) \delta(\epsilon - \mu) d\epsilon = G(\mu)$

Also note $\frac{df}{d\mu} = - \frac{df}{d\epsilon}$

$$\int_0^\infty G(\epsilon) \frac{df_0}{d\mu} d\epsilon \approx G(\mu) \int \frac{df_0}{d\mu} d\epsilon = - G(\mu) \int \frac{df_0}{d\epsilon} d\epsilon$$

$$= - G(\mu) f_0 \Big|_0^\infty = G(\mu) f_0(0) = G(\mu)$$

because $f(\infty) = 0, \quad f(0) = 1$

now

$$\frac{df_0}{dx} = \delta(\epsilon - \mu) \frac{d\mu}{dx}$$

$$J_n^x = - \left(\frac{d\mu}{dx} \right) \tau_c \int v_x^2 \delta(\epsilon - \mu) \mathcal{D}(\epsilon) d\epsilon$$

$$= - \frac{d\mu}{dx} \tau_c \frac{1}{3} \int v^2 \delta(\epsilon - \mu) \mathcal{D}(\epsilon) d\epsilon$$

$$= - \frac{d\mu}{dx} \tau_c \frac{2}{3m} \int \epsilon \delta(\epsilon - \mu) \mathcal{D}(\epsilon) d\epsilon$$

$$= - \frac{d\mu}{dx} \tau_c \frac{2}{3m} \mu \mathcal{D}(\mu)$$

$$= - \frac{d\mu}{dx} \tau_c \frac{n}{m}$$

now $\frac{d\mu}{dx} = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$

$$\frac{d\mu}{dx} = \left\{ \frac{2}{3} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \frac{1}{h^3} \right\} \frac{dn}{dx}$$

$$= \frac{2}{3} \cdot \frac{E_F}{h} \cdot \frac{dn}{dx}$$

$$J_n^x = - \left(\frac{2\tau_c}{3m} \right) E_F \frac{dn}{dx} = - \frac{1}{3} v_F^2 \tau_c \frac{dn}{dx} = -D \frac{dn}{dx}$$

$$D_{\text{quantum}} = \frac{1}{3} v_F^2 \tau_c$$

DC Electrical Conductivity

$$\frac{\partial f}{\partial t} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = - \frac{f-f_0}{\tau_c}$$

Steady state, $m\vec{a} = \vec{F} = e\vec{E}$ $\vec{a} = \frac{e\vec{E}}{m}$

$$\frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = - \frac{f-f_0}{\tau_c} - \vec{v} \cdot \frac{\partial f}{\partial \vec{x}}$$

$$\frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{v}} \frac{\partial f}{\partial \vec{E}} = - \frac{f-f_0}{\tau_c} \quad \frac{\partial f}{\partial \vec{x}} \frac{d\vec{x}}{d\vec{v}} + \frac{\partial f}{\partial \vec{v}} \frac{d\vec{v}}{d\vec{x}}$$

$$= \frac{e\vec{E}}{m} \cdot m\vec{v} \frac{\partial f}{\partial \vec{E}} = - \frac{f-f_0}{\tau_c} \quad \text{consider x-direction only}$$

$$f = f_0 - \tau_c e E v_x \frac{\partial f}{\partial E}$$

$$f' = f_0 - \tau_c e E v_x \frac{\partial f_0}{\partial E} = f_0 + \frac{\tau_c}{2} e E v_x f_0$$

Current $J = \langle e v_x f \rangle = \frac{\tau_c}{2} e^2 E \int v_x^2 f_0 D(E) dE = \frac{e^2 \tau_c n}{m} E = \sigma E$

$$\sigma = \frac{e^2 \tau_c n}{m}$$

linear response theory (transport theory)

$$\left[\frac{\partial f}{\partial t} + \frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{v}} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} = \frac{f-f_0}{\tau} \right]$$

AC conductivity

$$f = f_0 + f_1 \quad \frac{\partial f}{\partial t} = i\omega f$$

$$-i\omega f + \frac{eE}{m} \frac{\partial f}{\partial v} \frac{\partial f}{\partial E} = \frac{f-f_0}{\tau}$$

$$(1 + i\omega\tau) f = f_0 - \frac{eE}{m} \frac{v}{m} \frac{\partial f}{\partial E}$$

$$f = \frac{f_0}{(1 + i\omega\tau)} - \frac{eE v}{m m (1 + i\omega\tau)} \frac{\partial f_0}{\partial E}$$

$$\int v_x f dE d\vec{v} = \sigma(\omega) E$$

$$\sigma(\omega) = \frac{\sigma_0}{(1 + i\omega\tau_c)}$$

$$\sigma_0 = \frac{ne^2\tau_c}{m}$$

high frequency $\rightarrow \frac{ne^2}{i m \omega}$

$$D = \frac{\tau_c \tau}{m} = \frac{1}{3} \langle v^2 \rangle \tau_c = - \frac{\eta \tau_c}{m} \frac{d\mu}{dx}$$

next $\tau_c = \frac{l}{v}$

$$J_n = - \left(\frac{d\mu}{dx} \right) \left(\frac{l}{\tau_c} \right) \int \left(\frac{v_x^2}{v} \right) f_0 D(E) dE = \frac{1}{6} n$$

$\downarrow$
 $\frac{1}{3} n \bar{v}$

$$J_n^x = - \frac{1}{3} l \bar{v} \left(\frac{dn}{dx} \right)$$

M-1

Magnetoresistance (Hall effect)

Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \vec{\alpha} \cdot \frac{\partial f}{\partial \mathbf{v}} = - \frac{f - f_0}{\tau_c}$$

$\boxed{\mathbf{J} \parallel \mathbf{E}}$
not anymore
with magnetic
field.

Steady state $\frac{\partial f}{\partial t} = 0$ uniform system $\frac{\partial f}{\partial \mathbf{r}} = 0$

$$\vec{\alpha} = \frac{\vec{F}}{m} = \frac{e}{m} (\vec{E} + \vec{v} \times \vec{B})$$

We have $\frac{e}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = - \frac{f - f_0}{\tau_c}$

Only require linear response to $\vec{E}$ (but not in B)

$$\vec{E} \cdot \frac{e}{m} \frac{\partial f}{\partial \mathbf{v}} \rightarrow \frac{e}{m} \vec{E} \cdot \vec{v} m \frac{\partial f_0}{\partial \mathbf{v}} = e \vec{E} \cdot \vec{v} \frac{\partial f_0}{\partial \mathbf{v}}$$

$$f - f_0 = g \quad f = g + f_0$$

$$\frac{e}{m} (\vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{e}{m} (\vec{v} \times \vec{B}) \cdot \left(\frac{\partial g}{\partial \mathbf{v}} + \frac{\partial f_0}{\partial \mathbf{v}} \right)$$

Second term $\frac{e}{m} (\vec{v} \times \vec{B}) \cdot \vec{v} m \frac{\partial f_0}{\partial \mathbf{v}} = 0$

because $(\vec{v} \times \vec{B}) \cdot \vec{v} \equiv 0$

$$\therefore - e \vec{E} \cdot \vec{v} \frac{\partial f}{\partial \mathbf{v}} = \frac{g}{\tau_c} + \frac{e}{m} (\vec{v} \times \vec{B}) \cdot \frac{\partial g}{\partial \mathbf{v}}$$

M-2

for free electron $\vec{p} = m \vec{v} \quad \frac{1}{m} \frac{\partial g}{\partial \mathbf{v}} = \frac{\partial g}{\partial \mathbf{p}} = \frac{1}{\hbar} \frac{\partial g}{\partial \mathbf{k}}$

try following solution

$$g_{\mathbf{k}} = \left(- \frac{\partial f_0}{\partial \mathbf{E}} \right) \tau \vec{v}_{\mathbf{k}} \cdot e \vec{A}$$

$$\frac{\partial g}{\partial \mathbf{v}} = \left(- \frac{\partial f_0}{\partial \mathbf{E}} \right) \tau e \vec{A}$$

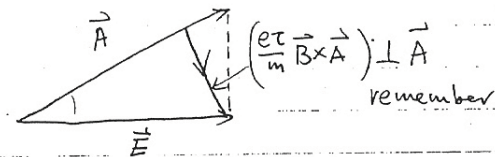
 $\vec{A}$ is an unknown vector, putting back to Boltzmann eq

$$- \vec{v} \cdot \vec{E} e \left(\frac{\partial f_0}{\partial \mathbf{E}} \right) = \left(- \frac{\partial f_0}{\partial \mathbf{E}} \right) \tau \vec{v}_{\mathbf{k}} \cdot e \vec{A} + \frac{e^2}{m} (\vec{v} \times \vec{B}) \cdot \tau \vec{A}$$

$$\text{or } \vec{v} \cdot \vec{E} = \vec{v} \cdot \vec{A} + \frac{e \tau}{m} (\vec{v} \times \vec{B}) \cdot \vec{A}$$

$$\text{or } \vec{v} \cdot \vec{E} = \vec{v} \cdot \vec{A} + \frac{e \tau}{m} (\vec{B} \times \vec{A}) \cdot \vec{v}$$

for arbitrary $\vec{v} \quad \vec{E} = \vec{A} + \frac{e \tau}{m} \vec{B} \times \vec{A}$

this is a vector equation that may be solved for $\vec{A}$ 

$$\vec{A} = \frac{\vec{E} - \frac{e \tau}{m} \vec{B} \times \vec{A}}{1 + \frac{e^2 \tau^2}{m^2} B^2} = \frac{\vec{E} - \frac{e \tau}{m} \vec{B} \times \vec{A}}{1 + \omega_c^2 \tau^2}$$

$$\omega_c = \text{frequency of cyclotron motion} = \frac{e B}{m}$$

$$\vec{E} = \vec{A} + \frac{e\tau}{m} \vec{B} \times \vec{A}$$

$$\vec{E} \cdot \vec{B} = \vec{A} \cdot \vec{B} \equiv 0 \quad \text{choose } \vec{E} \perp \vec{B}$$

$$\begin{aligned} \vec{B} \times \vec{E} &= \vec{B} \times \vec{A} + \frac{e\tau}{m} \vec{B} \times (\vec{B} \times \vec{A}) \\ &= \vec{B} \times \vec{A} + \frac{e\tau}{m} [(\vec{B} \cdot \vec{A}) \vec{B} - B^2 \vec{A}] \end{aligned}$$

now $\vec{E} = \vec{A} + \frac{e\tau}{m} \vec{B} \times \vec{A}$

$$\frac{e\tau}{m} (\vec{B} \times \vec{E}) = \frac{e\tau}{m} (\vec{B} \times \vec{A}) = \frac{e^2 \tau^2}{m^2} B^2 \vec{A}$$

$$\vec{E} - \frac{e\tau}{m} (\vec{B} \times \vec{E}) = \vec{A} \left(1 + \frac{e^2 \tau^2 B^2}{m^2} \right)$$

$$\vec{A} = \frac{\vec{E} - \frac{e\tau}{m} (\vec{B} \times \vec{E})}{1 + \frac{e^2 \tau^2 B^2}{m^2}} \neq$$

Method

$$\vec{A} = \vec{E} - \frac{e\tau}{m} \vec{B} \times \vec{A}$$

$$\vec{B} \cdot \vec{A} = \vec{B} \cdot \vec{E} \neq 0$$

$$\begin{aligned} \vec{B} \times \vec{A} &= \vec{B} \times \vec{E} - \frac{e\tau}{m} \vec{B} \times (\vec{B} \times \vec{A}) \quad \left(\vec{B} \times (\vec{B} \times \vec{A}) = (\vec{B} \cdot \vec{A}) \vec{B} - (\vec{B} \cdot \vec{B}) \vec{A} + (\vec{A} \cdot \vec{B}) \vec{B} \right) \\ &= \vec{B} \times \vec{E} + \frac{e\tau}{m} B^2 \vec{A} - \frac{e\tau}{m} (\vec{A} \cdot \vec{B}) \vec{B} \\ \vec{B} \times (\vec{B} \times \vec{A}) &= \vec{B} \times (\vec{B} \times \vec{E}) + \frac{e\tau}{m} \vec{B} \times (\vec{B} \times \vec{A}) \\ \vec{A} &= \vec{E} - \frac{e\tau}{m} \vec{B} \times \vec{E} - \frac{e^2 \tau^2}{m^2} \vec{A} \end{aligned}$$

$$\vec{A} \left(1 + \frac{e^2 \tau^2 B^2}{m^2} \right) = \vec{E} - \frac{e\tau}{m} \vec{B} \times \vec{E}$$

$$g_k = \left(- \frac{\partial f^0}{\partial \epsilon} \right) \tau v \cdot e \vec{A} = \left(- \frac{\partial f^0}{\partial \epsilon} \right) \frac{\tau e}{\hbar \omega_c \tau^2} \vec{v} \cdot \left(\vec{E} - \frac{e\tau}{m} \vec{B} \times \vec{E} \right)$$

$$J = \langle evf \rangle = \langle evf_0 \rangle + \langle evg \rangle$$

$$= \tau e \vec{A} \cdot \int \vec{v} \vec{v} \frac{\partial f^0}{\partial \epsilon} g(\epsilon) d\epsilon = \frac{\vec{A}}{\omega_c} \cdot \vec{A} = \frac{1}{\omega_c} \vec{A} \cdot \vec{A}$$

$$\vec{A} = \frac{1}{\omega_c} \vec{J}$$

$$\vec{A} = -\tau e \int \vec{v} \vec{v} \frac{\partial f^0}{\partial \epsilon} g(\epsilon) d\epsilon$$

$$\vec{E} = \frac{1}{\omega_c} \vec{J} + \frac{e\tau}{m} \frac{1}{\omega_c} \vec{B} \times \vec{J} = -S_s \vec{J} + \frac{e\tau}{m} S_o \vec{B} \times \vec{J}$$

$$\vec{J} = \omega_c \left(\frac{\vec{E} - \frac{e\tau}{m} \vec{B} \times \vec{E}}{1 + \frac{e^2 \tau^2 B^2}{m^2}} \right) = \frac{\omega_c}{1 + \omega_c^2 \tau^2} \left(\vec{E} - \frac{e\tau}{m} \vec{B} \times \vec{E} \right)$$

m-4

choose

$$\vec{E} = iE_x + jE_y \quad \vec{B} = kB$$

$$\vec{B} \times \vec{E} = \begin{vmatrix} i & j & k \\ 0 & 0 & B_z \\ E_x & E_y & E_z \end{vmatrix} = -iB E_y + jB E_x -$$

$$\vec{J} = \frac{\sigma_0}{1 + \omega_c^2 \tau_0^2} \left[(iE_x + jE_y) + kE_z - iB E_y \frac{e\tau}{m} + jB E_x \frac{e\tau}{m} \right]$$

$$= \frac{1}{\sigma} \vec{E} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\sigma_{xx} = \frac{\sigma_0}{1 + \omega_c^2 \tau_0^2} = \sigma_{yy} = \sigma_{zz}$$

$$\sigma_{xy} = -\omega_c \tau_0 \sigma_{xx} \quad \sigma_{yx} = -\sigma_{xy}$$

$$\vec{R} = (\vec{\sigma})^{-1}$$

$$\sigma_{xy} = \frac{\sigma_{yx}}{\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2} = \frac{\omega_c \tau_0 \sigma_{xx} (1 + \omega_c^2 \tau_0^2)}{\sigma_0^2 + \sigma_0^2 \omega_c^2 \tau_0^2} = \frac{\omega_c \tau_0}{\sigma_0} = \frac{qB\tau}{m n e} = \frac{B}{n e c}$$

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_{xx}}{\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2} = \frac{1}{(1 + \omega_c^2 \tau_0^2)} = \frac{1}{\sigma_0}$$

$$\sigma_{zz} = \frac{1}{\sigma_{zz}}$$

Hall effect
↑

1. Ratio of thermal to electrical conductivity. Show for a classical gas of particles of charge q that

$$K/\tau\sigma = 3/2q^2 \quad \text{or} \quad K/T\sigma = 3k_B^2/2q^2 \quad (124)$$

in conventional units for K and T . This is known as the Wiedemann-Franz ratio.

2. Boltzmann equation and thermal conductivity. Consider a medium with temperature gradient dt/dx . The particle concentration is constant. (a) Employ the Boltzmann transport equation in the relaxation time approximation to find the first order nonequilibrium classical distribution:

$$f \approx f_0 - v_x \tau_c \left(-\frac{3}{2\tau} + \frac{v^2}{\tau^2} \right) f_0 \frac{d\tau}{dx} \quad (125)$$

(b) Show that the energy flux in the x direction is

$$J_u = - \left(\frac{d\tau}{dx} \right) \tau_c \int v_x^2 \left(-\frac{3v}{2\tau} + \frac{v^2}{\tau^2} \right) f_0 \mathcal{D}(v) dv \quad (126)$$

where $\tau_c^{-2} = 2\pi/3m$. (c) Evaluate the integral to obtain for the thermal conductivity $K = 5n\tau_c m$.