

The University of Wollongong

School of Engineering Physics

Spring Session Examination

November 2008

PHYS385–STATISTICAL MECHANICS

Time allowed – 2.0 hours

Total number of questions – 4

Answer ALL 4 questions

All questions are of equal value

1. The multiplicity function for a system of N_1 spins with spin excess $2s_1$ is $g(N_1, s_1) = g_1(0) \exp(-2s_1^2/N_1)$. Consider two spin systems, (N_1, s_1) and (N_2, s_2) . These two systems are brought into thermal contact to form a combined system (N, s) with N and s being held constant.

(i) What is the number of states accessible to the combined system when the spin excess of the first system is $2s_1$?

(ii) Find the most probable value of s_1 (denoted as $\hat{s}_1$).

(iii) Let $N_1 = N_2 = 10^{24}$ and let s_1 deviate from $\hat{s}_1$ by an amount $\delta = 10^{13}$, (or $\delta/N_1 = 10^{-11}$), determine the order of magnitude of the ratio

$$\frac{g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_2 - \delta)}{(g_1g_2)_{max}}.$$

(iv) Show that entropy for the above system is given by $\ln(g_1g_2)_{max}$ due to the sharpness of the multiplicity function, (hint: show that $\sum_{s_1} g_1g_2 = A(g_1g_2)_{max}$ and the contribution of A to the entropy can be ignored).

2. **(A)** A one-dimensional harmonic oscillator has an infinite series of equally spaced energy states, with $\epsilon_s = s\hbar\omega$, where s is a positive integer or zero, and ω is the classical frequency of the oscillator.

(i) Determine the partition function and mean occupancy as a function of temperature and frequency.

(ii) Determine the Helmholtz free energy and the entropy of the system.

(B) The rotational energy of molecules is quantised and energy levels of a diatomic molecule are given as

$$\epsilon_j = j(j+1)\epsilon_0$$

where j is any positive integer or zero, ϵ_0 is a constant. The number of states in level j is $g_j = 2j + 1$.

(iii) Calculate the partition function Z_R for the rotational states of one molecule.

(iv) Evaluate Z_R in the high temperature limit, $\tau \gg \epsilon_0$, then determine energy U and the heat capacity C in the same limit.

3. The partition function Z_N for a system of N independent particles is given as $Z_N = (n_q V)^N / N!$.

(i) Find the Gibbs sum $\mathcal{Z}$ and mean occupancy $\langle N \rangle$ for an ideal gas of identical atoms.

(ii) Show that the probability there are N atoms in the gas in volume V in diffusive contact with a reservoir is

$$P(N) = \frac{\langle N \rangle^N \exp(-\langle N \rangle)}{N!}.$$

(iii) Calculate $\langle (N - \langle N \rangle)^2 \rangle$.

(iv) Suggest a simple experimental procedure for the determination of $\langle N \rangle$.

(v) A donor level in a semiconductor can be either empty (ionised) or occupied by an electron with spin-up or spin-down. When the level is occupied, its binding energy is $-I$. Calculate the grand partition function of the donor atom and the probability that the donor is ionised.

4. (i) Electrons are described by the Fermi-Dirac distribution functions. Calculate the Fermi energy of an electron gas.

(ii) Calculate the electronic heat capacity at low temperature.

(iii) Show that for a Fermi gas, the fluctuation is given as,

$$\langle (\Delta N)^2 \rangle = \langle N \rangle (1 - \langle N \rangle),$$

where $\langle N \rangle$ is the average number of fermions in the orbital. Show that the fluctuation vanishes for orbitals with energy deep enough below the Fermi energy.

(iv) Explain why the chemical potential of a Bose gas must be always negative.

(v) Calculate the number of particles in all excited states in a Bose gas,

$$N_e = \int_0^\infty D(\epsilon) f_{BE}(\epsilon) d\epsilon.$$

Define the Einstein condensation temperature τ_E . Plot your result of N_e and the number of particles in the ground state N_0 versus τ/τ_E .

USEFUL EQUATIONS:

$$Z = \sum_s e^{-\epsilon_s/\tau}; \quad F = -\tau \ln(Z); \quad \sigma(N, U, V) = \ln(g(N, U, V))$$

$$\mathcal{Z} = \sum_{N,S} \lambda^N e^{-\epsilon_s(N)/\tau}; \quad \lambda = e^{\mu/\tau}; \quad \langle N \rangle = \lambda \frac{\partial \ln \mathcal{Z}}{\partial \lambda}$$

$$D(\epsilon) = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\epsilon}; \quad n_Q = \left(\frac{m\tau}{2\pi\hbar^2} \right)^{3/2}$$

$$f(v) = 4\pi \left(\frac{m}{2\pi\tau} \right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2\tau}\right)$$

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V; \quad \mu = \left(\frac{\partial F}{\partial N}\right)_{\tau,V}; \quad c_V = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V$$

$$U = \left(\mu\tau \frac{\partial}{\partial \mu} + \tau^2 \frac{\partial}{\partial \tau}\right) \ln \mathcal{Z}; \quad U = \tau^2 \frac{\partial}{\partial \tau} \ln Z$$

$$F_N - F_S = \frac{B^2}{2\mu_0}$$

$$n_e = n_c \exp\left(-\frac{\epsilon_c - \mu}{\tau}\right); \quad n_h = n_v \exp\left(\frac{\epsilon_v - \mu}{\tau}\right)$$

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}; \quad \int_0^\infty dx e^{-x^2} = \frac{\sqrt{\pi}}{2}$$

$$P = \tau \left(\frac{\partial \sigma}{\partial V}\right)_{U,N} = \left(\frac{\partial U}{\partial V}\right)_{\sigma,N} = \left(\frac{\partial F}{\partial V}\right)_{\tau,N}$$

$$f_{FD} = \frac{1}{e^{(\epsilon-\mu)/\tau} + 1}; \quad f_{BE} = \frac{1}{e^{(\epsilon-\mu)/\tau} - 1}$$

$$U_B = -\frac{1}{2} \mathbf{M} \cdot \mathbf{B}$$

$$n_e n_h = n_i^2$$

$$\int_0^\infty dx \frac{\sqrt{x}}{e^x - 1} = 1.306\pi^{1/2}$$

$$Z_1 = n_Q V, \quad Z_N = \frac{(n_Q V)^N}{N!}$$