

MECH321

Dynamics of Engineering System

Week 4 (Chapter 6)

1. Basic electric circuit theories

2. Mathematical Modeling of Passive Circuits

3. Complex Impedance Approach

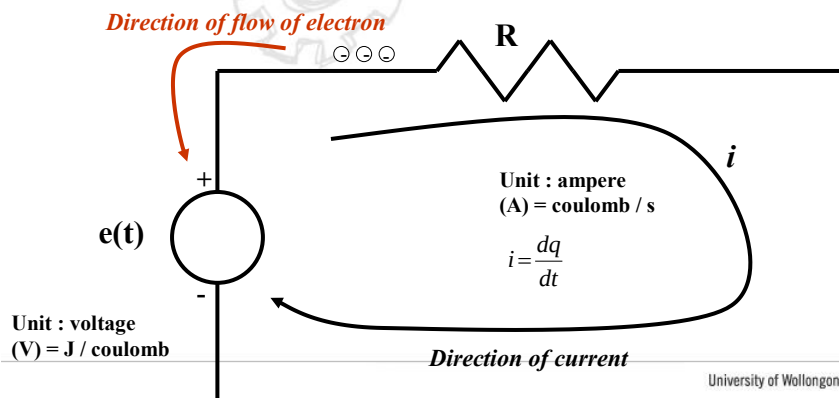
4. Mechanical – Electrical analogy

5. Modelling of Active Circuits:
Operational – Amplifier Circuits

Basic Electric Circuit Theories

Two important variables in electrical circuit analysis:

- **Voltage, e** : represents the potential energy in the circuit
- **Current, i** : represents rate of charge q with respect of time. It can be positive or negative, depending upon the direction in which the electrons flow.



Basic passive elements

- Resistance: unit ohm (Ω) = volt / ampere

$$R = \frac{e_R}{i} \quad \text{Ohm's law}$$

$$e_R = Ri$$

- Capacitance: unit farad (F) = coulomb / volt

$$C = \frac{q}{e_C} \longrightarrow e_C(t) = \frac{1}{C} \int_0^t i dt + e_C(0)$$

- Inductance: unit Henry (H) = V.s / A

$$L = \frac{e_L}{di/dt} \longrightarrow e_L(t) = L \frac{di_L}{dt}$$

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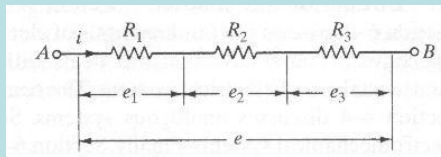
Ohm's law

Using Ohm's law the following rules governing electrical circuits can be derived:

- Series circuits

$$e_{total} = e_1 + e_2 + e_3 + \dots$$

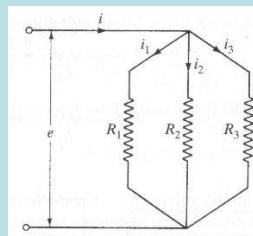
$$R_{total} = R_1 + R_2 + \dots$$



- Parallel circuits

$$i_{total} = i_1 + i_2 + \dots$$

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots$$

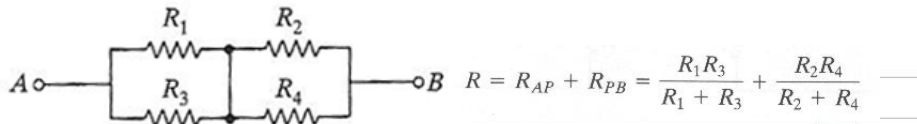
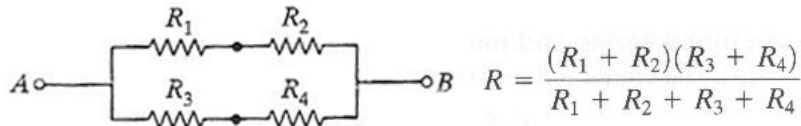
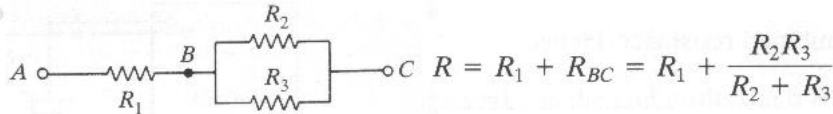


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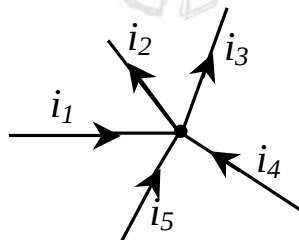
Ohm's law

Using the series and parallel circuit theories, combined resistance of series/parallel resistors can be calculated:



1st Kirchhoff's law

1st Kirchhoff's Law "Current law (node law)":
the sum of all currents entering (+) and leaving (-) a node is zero

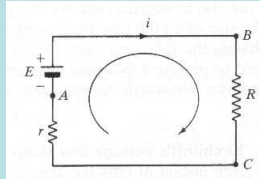


$$i_1 - i_2 - i_3 + i_4 + i_5 = 0$$

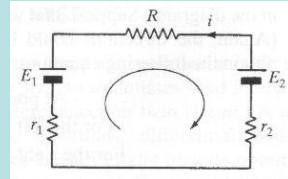
2nd Kirchoff's law

2nd Kirchoff's Law "**Voltage law (loop law)**": the sum of the voltages around any loop is zero (a rise in voltage (+); a drop in voltage (-))

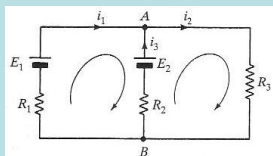
One-loop circuit



$$E - iR - ir = 0$$



$$E_1 - iR - E_2 - ir_1 - ir_2 = 0$$



Two-loop circuit

Left loop: $E_1 - E_2 + i_3 R_2 - i_1 R_1 = 0$

Right loop: $E_2 - i_2 R_3 - i_3 R_2 = 0$

Mathematical Modeling of Passive Circuits:

LR circuit

Transfer Function: $\frac{E_o(s)}{E_i(s)} = G(s)$

From Kirchoff's voltage law around a loop

$$L \frac{di}{dt} + Ri = E$$

Taking Laplace transform of both sides

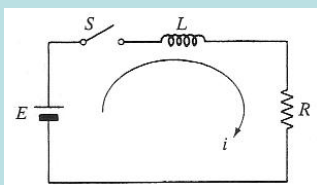
$$L[sI(s) - i(0)] + RI(s) = \frac{E}{s} \quad \text{Step input}$$

$$i(0) = 0 \quad (Ls + R)I(s) = \frac{E}{s}$$

$$I(s) = \frac{E}{s(Ls + R)} = \frac{E}{R} \left[\frac{1}{s} - \frac{1}{s + (R/L)} \right]$$

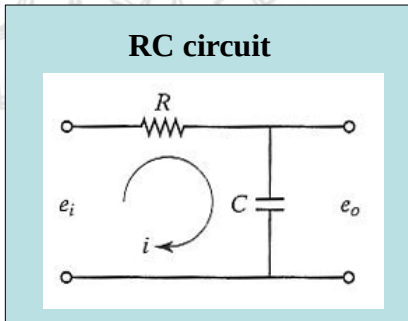
$$\therefore i(t) = \frac{E}{R} (1 - e^{-(R/L)t})$$

LR circuit



Mathematical Modeling of Passive Circuits:

RC circuit



From Kirchoff' voltage law around a loop

$$\text{Loop 1} \longrightarrow Ri + \frac{1}{C} \int idt = e_i$$

$$\text{Loop 2} \longrightarrow \frac{1}{C} \int idt = e_o$$

Taking Laplace transform of both equations

$$RI(s) + \frac{1}{C} \frac{1}{s} I(s) = E_i(s)$$

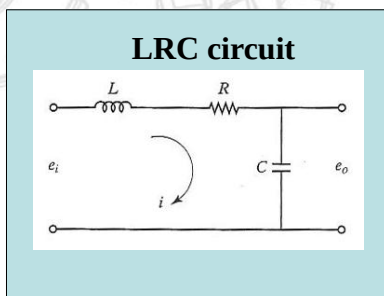
$$\frac{1}{C} \frac{1}{s} I(s) = E_o(s)$$

If e_o is the output and e_i is the input, the transfer function of the system is

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{RCs + 1}$$

Mathematical Modeling of Passive Circuits:

LRC circuit



From Kirchoff' voltage law around a loop

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int idt = e_i$$

$$\frac{1}{C} \int idt = e_o$$

Taking Laplace transform of both equations

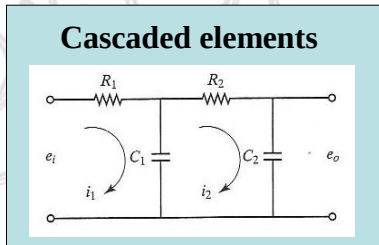
$$LsI(s) + RI(s) + \frac{1}{C} \frac{1}{s} I(s) = E_i(s)$$

$$\frac{1}{C} \frac{1}{s} I(s) = E_o(s)$$

If e_o is the output and e_i is the input, the transfer function of the system is

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Cascaded elements



From Kirchoff' voltage law around a loop

$$R_1 i_1 + \frac{1}{C_1} \int (i_1 - i_2) dt = e_i$$

$$R_2 i_2 + \frac{1}{C_1} \int (i_2 - i_1) dt + \frac{1}{C_2} \int i_2 dt = 0$$

$$\frac{1}{C_2} \int i_2 dt = e_o$$

Taking Laplace transform of the equations

$$R_1 I_1(s) + \frac{1}{C_1 s} [I_1(s) - I_2(s)] = E_i(s)$$

$$R_2 I_2(s) + \frac{1}{C_1 s} [I_2(s) - I_1(s)] + \frac{1}{C_2 s} I_2(s) = 0$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s)$$

After eliminating $I_1(s)$ and $I_2(s)$, the transfer function of the system is obtained.

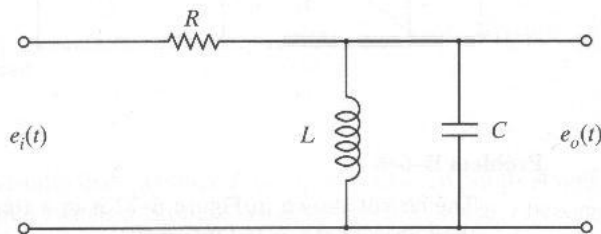
$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2) s + 1}$$

Tutorial

Example 1 (B-6-9)

Obtain the transfer function $E_o(s)/E_i(s)$ of the circuit below.



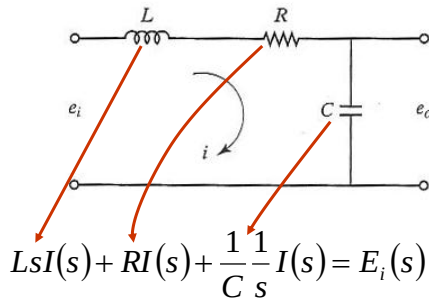
(solution will be done in the lecture)

Complex Impedance Approach

Complex Impedance: $Z(s) = \frac{E(s)}{I(s)}$

Using this approach transfer functions of simple circuits are obtained *directly* as the Laplace-transformed equations.

Consider again LRC circuit:



Complex impedance of:
an inductor

$$Z_L = Ls$$

A resistor

$$Z_R = R$$

A capacitor

$$Z_C = \frac{1}{Cs}$$

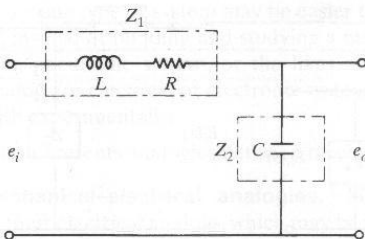
$$\frac{1}{C} \frac{1}{s} I(s) = E_o(s)$$

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Complex Impedance Approach

Using complex impedances, the Laplace-transformed Equations are:



$$Z_L I(s) + Z_R I(s) + Z_C I(s) = E_i(s)$$

$$Z_C I(s) = E_o(s)$$

These equations will give the transfer function

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{Z_C}{Z_L + Z_R + Z_C} = \frac{1}{LCs^2 + RCs + 1}$$

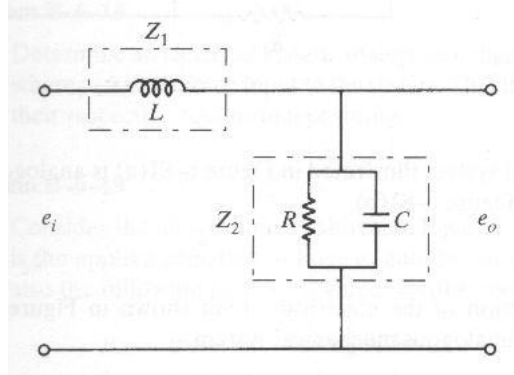
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Classroom exercise

Example 2 (B-6-12)

Obtain the transfer function $E_o(s)/E_i(s)$ of the circuit below using the complex-impedance method.



(solution will be done in the lecture)

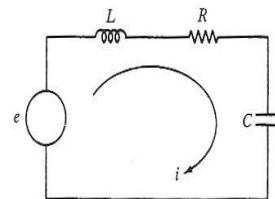
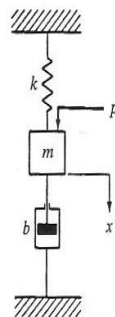
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Mechanical-electrical analogies: Force-voltage analogy

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = e$$

$$i = \frac{dq}{dt} = \dot{q}$$



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = e$$

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Force-voltage analogy

$$m\ddot{x} + b\dot{x} + kx = p$$

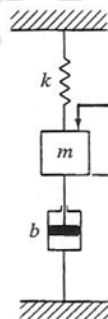
$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = e$$

TABLE 6-1 Force-Voltage Analogy

Mechanical Systems	Electrical Systems
Force p (torque T)	Voltage e
Mass m (moment of inertia J)	Inductance L
Viscous-friction coefficient b	Resistance R
Spring constant k	Reciprocal of capacitance, $1/C$
Displacement x (angular displacement θ)	Charge q
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Current i

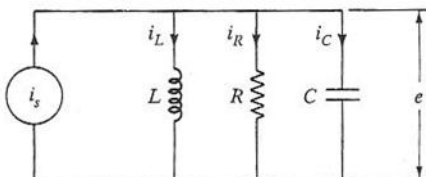
Mechanical-electrical analogies: Force-current analogy



$$i_L + i_R + i_C = i_s$$

$$i_L = \frac{1}{L} \int e dt, i_R = \frac{e}{R}, i_C = C \frac{de}{dt}$$

$$\text{so } \frac{1}{L} \int e dt + \frac{e}{R} + C \frac{de}{dt} = i_s$$



Mechanical-electrical analogies: Force-current analogy

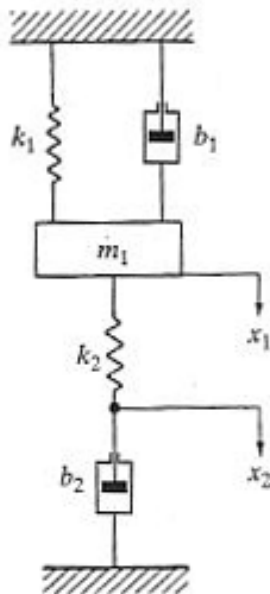
$$\frac{1}{L} \int e dt + \frac{e}{R} + C \frac{de}{dt} = i_s$$

$$m\ddot{x} + b\dot{x} + kx = P \quad m\dot{v} + bv + k \int v(\tau) d\tau = P$$

TABLE 6-2 Force-Current Analogy

Mechanical Systems	Electrical Systems
Force p (torque T)	Current i
Mass m (moment of inertia J)	Capacitance C
Viscous-friction coefficient b	Reciprocal of resistance, $1/R$
Spring constant k	Reciprocal of inductance, $1/L$
Displacement x (angular displacement θ)	Magnetic flux linkage ψ
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Voltage e

A-6-16 (p.304): Find an electric analogy

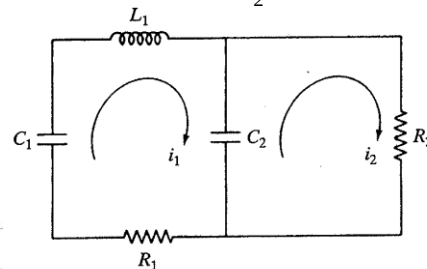


$$m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0$$

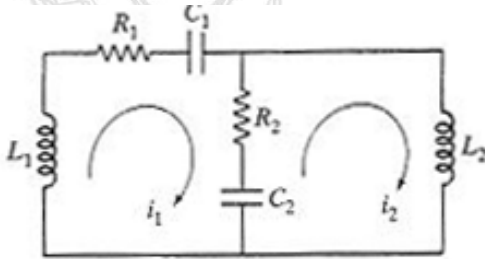
$$b_2 \dot{x}_2 + k_2 (x_2 - x_1) = 0$$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int i_1 dt + \frac{1}{C_2} \int (i_1 - i_2) dt = 0$$

$$R_2 i_2 + \frac{1}{C_2} \int (i_2 - i_1) dt = 0$$



A-6-17: Find a mechanical analogy

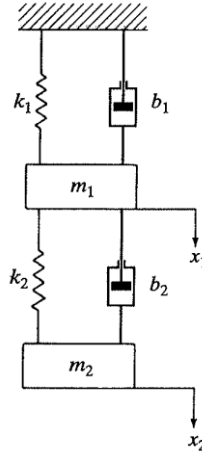


$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int i_1 dt + \frac{1}{C_2} \int (i_1 - i_2) dt + R_2 (i_1 - i_2) = 0$$

$$L_2 \frac{di_2}{dt} + R_2 (i_2 - i_1) + \frac{1}{C_2} \int (i_2 - i_1) dt = 0$$

$$m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) + b_2 (\dot{x}_1 - \dot{x}_2) = 0$$

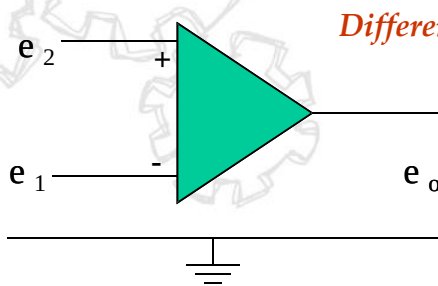
$$m_2 \ddot{x}_2 + k_2 (x_1 - x_2) + b_2 (\dot{x}_2 - \dot{x}_1) = 0$$



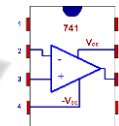
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Modelling of Active Circuits: Operational Amplifier (op-amp) Circuits



Differential amplifier



$$e_o = K(e_2 - e_1) = -K(e_1 - e_2)$$

Very large gain $\approx 10^5$

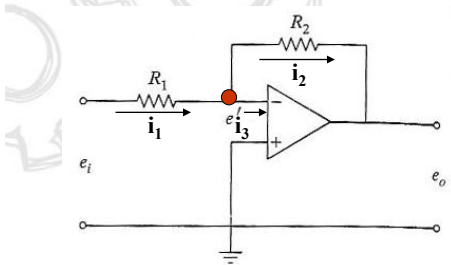
Input voltages difference

Op-amp has a large input impedance, so it draws a negligible current.

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Op-Amp Amplifier: proportional amplifier



Applying Kirchoff's node law

$$i_1 = i_2 + i_3$$

But $i_3 \approx 0$,

Op-amp has a large input impedance, so it draws a negligible current.

$$\frac{e_i - e'}{R_1} + \frac{e_o - e'}{R_2} = 0$$

$$e_o = -Ke'$$

$$\frac{e_o}{e_i} = \frac{-R_2 / R_1}{1 + \frac{R_2}{K R_1}} \cong -\frac{R_2}{R_1}$$

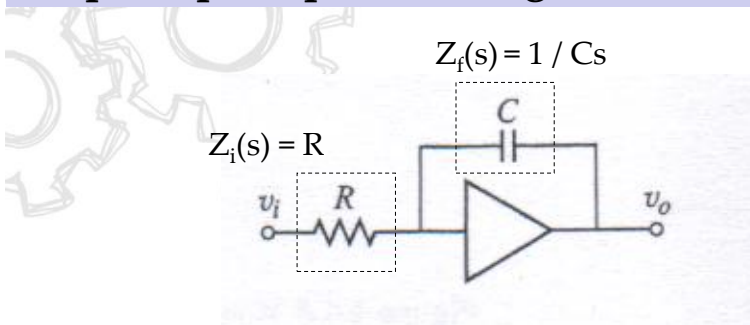
0

Because K is very large

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Op-Amp Amplifier: integrator



$$\text{The transfer function, } G(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} = -\frac{1}{RCs}$$

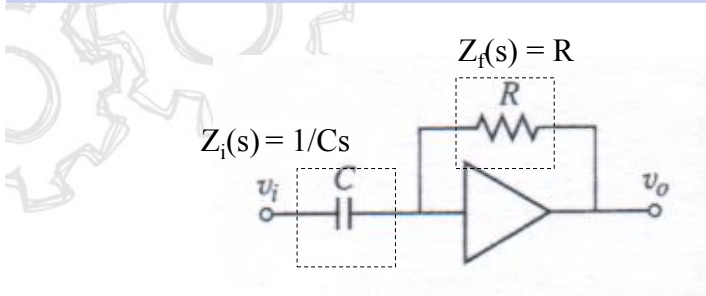
In the time domain

$$v_o(t) = -\frac{1}{RC} \int_0^t v_i dt$$

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Op-Amp Amplifier: differentiator



The transfer function, $G(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} = -RCs$

In the time domain,

$$v_o(t) = -RC \frac{dv_i(t)}{dt}$$

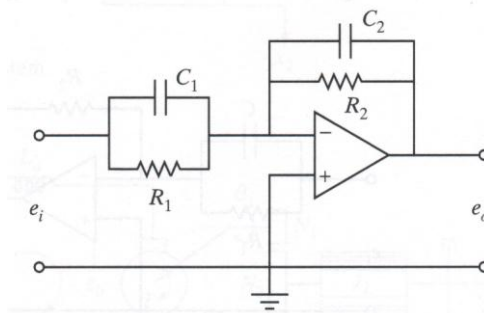
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Classroom exercise

Example 3 (B-6-25)

Obtain the transfer function $E_o(s)/E_i(s)$ of the following operational-amplifier shown below.



(solution will be done in the lecture)

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