

MECH419/928

Finite Element Methods in Engineering / Finite Element Techniques in Mechanical Engineering

Week 5 & 6 (chapter 8):

- Strain – displacement relations

- Plane stress

- Plane strain

- Axisymmetric solid



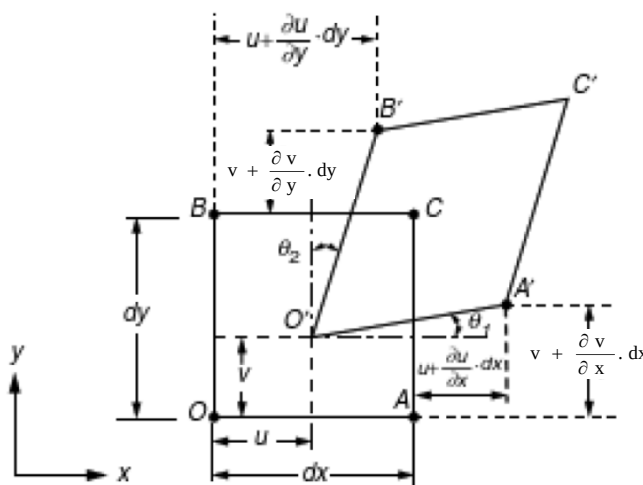
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Strain – displacement equations

- The strains can be expressed in terms of the displacements u , v and w .



$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u}{\partial x} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} \\ \epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \epsilon_{zz} &= \frac{\partial w}{\partial z} \\ \epsilon_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \epsilon_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\end{aligned}$$

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Plane stress

- The stress – strain relationships become

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

[D]

- In plane stress case, the component of strain in the z direction is not zero

$$\varepsilon_{zz} = -\frac{\nu}{E} [\sigma_{xx} + \sigma_{yy}] \quad \varepsilon_{yz} = \varepsilon_{zx} = 0$$

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Plane stress using simplex triangular element

- For a two-dimensional triangular (simplex) element, the interpolation model for each components, namely u and v are

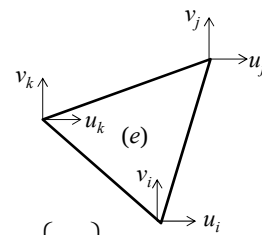
$$u^{(e)} = N_i u_i + N_j u_j + N_k u_k$$

$$v^{(e)} = N_i v_i + N_j v_j + N_k v_k$$

In matrix form

$$\vec{U} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N] \vec{Q}^{(e)}$$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \quad \vec{Q}^{(e)} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix}$$



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Plane stress using simplex triangular element

- The strain state in matrix form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} = [B] \bar{Q}^{(e)} \quad [B] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N]$$

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Plane stress using simplex triangular element

- The shape functions of triangular simplex element are

$$\left. \begin{aligned} N_i &= \frac{1}{2A}(a_i + b_i x + c_i y) \\ N_j &= \frac{1}{2A}(a_j + b_j x + c_j y) \\ N_k &= \frac{1}{2A}(a_k + b_k x + c_k y) \end{aligned} \right\} [B] = \frac{1}{2A} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix}$$

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The strain energy, π

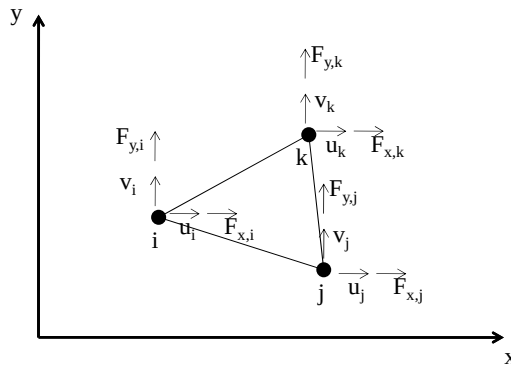
$$\begin{aligned}
 \pi^{(e)} &= \int_V \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} \, dv \\
 &= \frac{1}{2} \int_V ([\mathbf{D}][\boldsymbol{\varepsilon}])^T [\boldsymbol{\varepsilon}] \, dv \\
 &= \frac{1}{2} \int_V [\boldsymbol{\varepsilon}]^T [\mathbf{D}]^T [\boldsymbol{\varepsilon}] \, dv \\
 &= \frac{1}{2} \int_V [\boldsymbol{\varepsilon}]^T [\mathbf{D}] [\boldsymbol{\varepsilon}] \, dv
 \end{aligned}$$

Note that: $([\mathbf{D}][\boldsymbol{\varepsilon}])^T = [\boldsymbol{\varepsilon}]^T [\mathbf{D}]^T$, $[\mathbf{D}]^T = [\mathbf{D}]$ and $[\boldsymbol{\varepsilon}] = [\mathbf{B}] \bar{\mathbf{Q}}$

$$\pi^{(e)} = \frac{1}{2} \int_V \bar{\mathbf{Q}}^T [\mathbf{B}]^T [\mathbf{D}] [\mathbf{B}] \bar{\mathbf{Q}} \, dv$$

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The work, W



$$\begin{aligned}
 W^{(e)} &= u_i F_{x,i} + v_i F_{y,i} + u_j F_{x,j} + v_j F_{y,j} + u_k F_{x,k} + v_k F_{y,k} \\
 &= \bar{\mathbf{Q}}^T [\mathbf{F}]
 \end{aligned}$$

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The potential energy, I

$$\begin{aligned} I^{(e)} &= \pi^{(e)} - W^{(e)} \\ &= \frac{1}{2} \int_V \bar{Q}^T [B]^T [D] [B] \bar{Q} dV - \bar{Q}^T [F] \end{aligned}$$

Minimizing the potential energy, I with respect to the nodal displacements (the unknowns),

$$\left[\frac{\partial I}{\partial Q} \right] = \iiint_V ([B]^T [D] [B] \bar{Q}) dV - [F] = 0$$

$$V[B]^T [D][B]\bar{Q} - F = 0$$

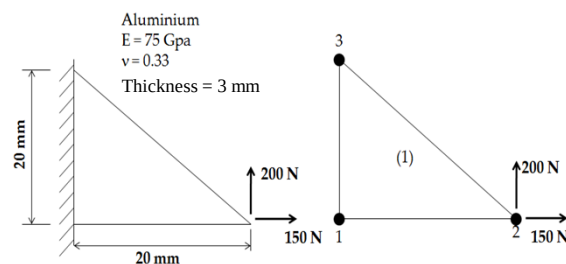
- The stiffness (characteristic) matrix is

$$[K] = V [B]^T [D] [B]$$

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Plane stress example

- A thin plate is supported and loaded as shown. Calculate:
 - Maximum deformation at the plate
 - determine if the plate deforms plastically under this load. and
 - Change in the plate thickness



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Plane strain

- The stress – strain relationships become

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

[D]

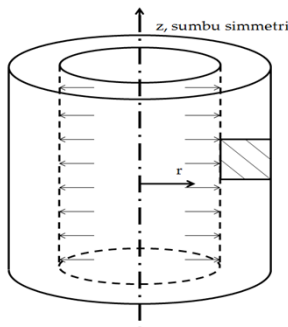
- In plane stress strain, the component of stress in the z direction is not zero

$$\sigma_{zz} = \nu [\sigma_{xx} + \sigma_{yy}] \quad \sigma_{yz} = \sigma_{zx} = 0$$

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Axisymmetric solid

- For axisymmetrical solids, $\varepsilon_{r\theta} = \varepsilon_{z\theta} = 0$



$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 \\ -\nu & 1 & -\nu & 0 \\ -\nu & -\nu & 1 & 0 \\ 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix}$$

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rz} \end{Bmatrix}$$

[D]

- subscripts r , θ , and z denote the radial, tangential, and axial directions

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Axisymmetric solid

- For a two-dimensional triangular (simplex) element, the interpolation model for each components, namely u and v are

$$u^{(e)} = N_i u_i + N_j u_j + N_k u_k$$

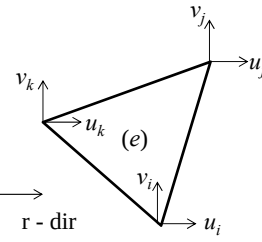
$$v^{(e)} = N_i v_i + N_j v_j + N_k v_k$$

In matrix form

$$\bar{U} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N] \bar{Q}^{(e)}$$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \bar{Q}^{(e)} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix}$$

Axis of rotation
↑ z - axis



Note that:
 v is the velocity in z-dir
 u is velocity in r-dir

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Axisymmetric solid using simplex triangular element

- The strain state in matrix form

$$\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & \frac{\partial N_j}{\partial r} & 0 & \frac{\partial N_k}{\partial r} & 0 \\ \frac{N_i}{r} & 0 & \frac{N_j}{r} & 0 & \frac{N_k}{r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_j}{\partial z} & 0 & \frac{\partial N_k}{\partial z} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial z} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial z} & \frac{\partial N_k}{\partial r} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix}$$

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Axisymmetric solid using simplex triangular element

$$\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rz} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ \frac{2AN_i}{r} & 0 & \frac{2AN_j}{r} & 0 & \frac{2AN_k}{r} & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix}$$

where $[B]$

$$r = \frac{r_1 + r_2 + r_3}{3}$$

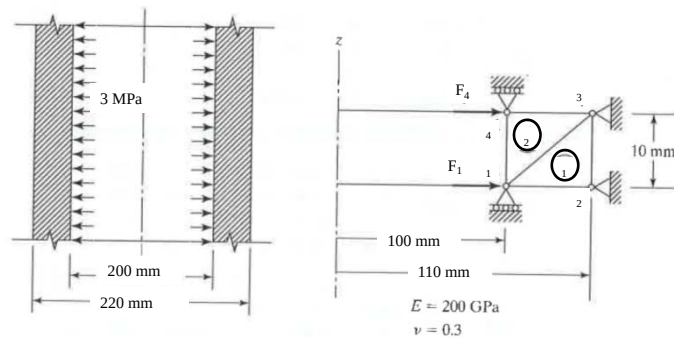
- The stiffness matrix is

$$[K] = \iiint_V [B]^T [D] [B] dV = V [B]^T [D] [B] = 2 \pi \bar{r} A [B]^T [D] [B]$$

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Axisymmetric example

A long cylinder of inside diameter 200 mm and outside diameter 220 mm snugly fits in a hole over its full length. The cylinder is then subjected to an internal pressure of 3 MPa. Using two elements on the 10 mm length shown, find the displacements at the inner radius.



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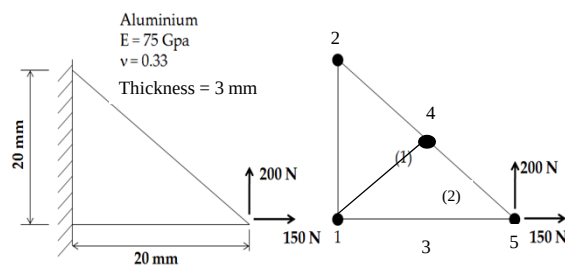
Tutorial Questions (week 5 & 6)

Q1. A thin plate is supported and loaded as shown using a combined quadrilateral and triangular elements. Calculate:

- (i) Maximum deformation at the plate
- (ii) Change in the plate thickness

and

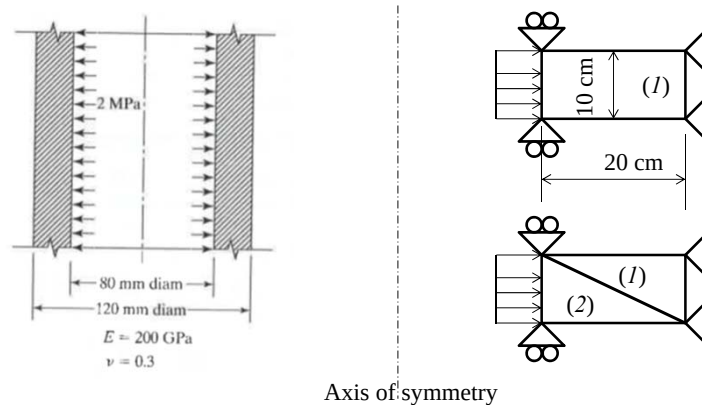
- (iii) determine if the plate deforms plastically under this load.



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Tutorial Questions (week 5 & 6)

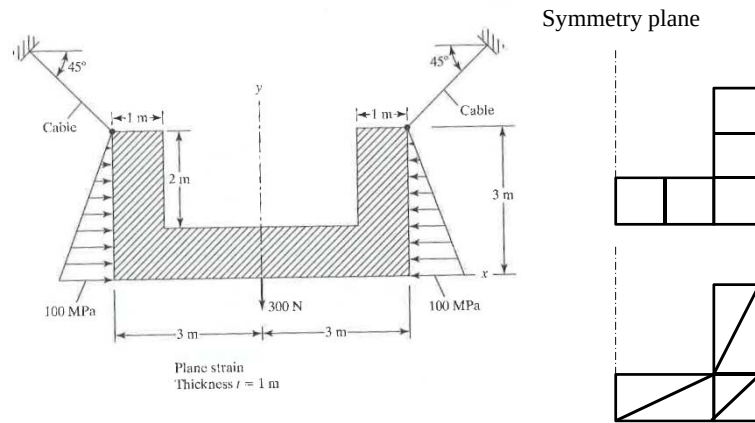
Q2. A long cylinder of inside diameter 200 mm and outside diameter 220 mm snugly fits in a hole over its full length. The cylinder is then subjected to an internal pressure of 3 MPa. Using one (quad) element and two (triangular) elements on the 10 mm length shown, find the displacements at the inner radius.



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Tutorial Questions (week 5 & 6)

Q3. A structure in plane strain is shown below. The structure and loading is symmetric with respect to the y-axis. Find the nodal displacement and the stress distribution in the body. Use five quadrilateral elements and six triangular elements.



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