

Computational Fluid Dynamics – Lecture 1

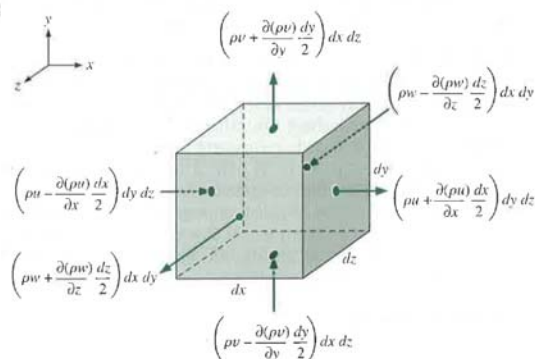
- The Governing Equations of Fluid Dynamics
- Finite Difference Discretisation
- 1-D flow solution
- MATLAB coding

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Continuity equation - conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$



The inflow or outflow of mass through each face of the differential control volume; the blue dots indicate the center of each face.

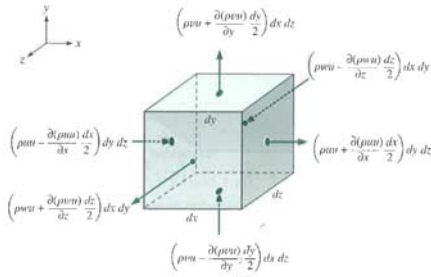
(Cengel & Cimbala, 2010)

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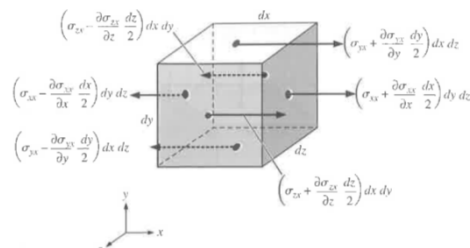


Conservation of Momentum

$$\begin{aligned}\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho vu)}{\partial z} &= \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} &= \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \\ \frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} &= \rho g_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\end{aligned}$$



Inflow and outflow of the x-component of linear momentum through each face of an infinitesimal control volume



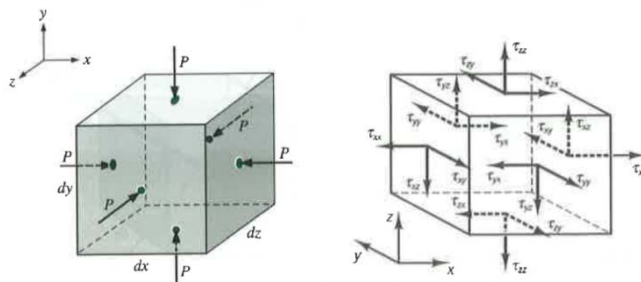
The surface forces acting in the x-direction on each face of the infinitesimal control volume

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(Cengel & Cimbala, 2010)

Constitutive Equations



$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \quad \text{Viscous stress Tensor, } \tau_{ij}$$

For Newtonian fluid

$$\tau_{ij} = \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & 2\mu \frac{\partial v}{\partial y} & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix}$$

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Continuity Equation & Navier – Stokes Equations



FIGURE 9-42
The Navier-Stokes equation is the cornerstone of fluid mechanics.

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

x-component of the incompressible Navier-Stokes equation:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} = -\frac{\partial p}{\partial x} + \rho g_x + \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) + \mu \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} \right)$$

y-component of the incompressible Navier-Stokes equation:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} = -\frac{\partial p}{\partial y} + \rho g_y + \mu \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) + \mu \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial z} \right)$$

z-component of the incompressible Navier-Stokes equation:

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = -\frac{\partial p}{\partial z} + \rho g_z + \mu \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} \right) + \mu \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial z} \right)$$

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Simplified – general differential form of the governing equations

It is clear that there are some significant commonalities between the various equations. We can introduce a general variable ϕ the conservative form of all fluid flow equations, including equations for scalar quantities such as temperature and pollutant concentration etc, the following form of the general equation can be written

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u_j \phi)}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi}{\partial x_j} \right) + q_\phi \quad j=1, 2, 3$$

$$\phi = 1 \quad q_\phi = 0 \quad \text{continuity equation}$$

$$\phi = u \quad q_\phi = -\frac{\partial p}{\partial x} + \rho g_x \quad x\text{-momentum equation}$$

$$\phi = v \quad q_\phi = -\frac{\partial p}{\partial y} + \rho g_y \quad y\text{-momentum equation}$$

$$\phi = w \quad q_\phi = -\frac{\partial p}{\partial z} + \rho g_z \quad z\text{-momentum equation}$$

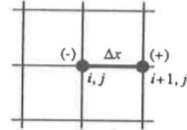
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Finite Difference Discretisation

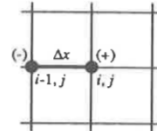
First-order
forward
difference
with respect
to x

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x}$$



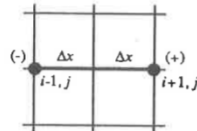
First-order
rearward
difference
with respect
to x

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x}$$



Second-order
central
difference
with respect
to x

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j} + u_{i-1,j}}{2\Delta x}$$



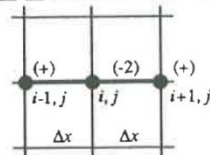
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Finite Difference Discretisation

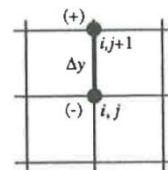
Second-order
central
second
difference
with respect
to x

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$



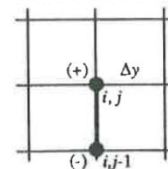
First-order
forward
difference
with respect
to y

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y}$$



First-order
rearward
difference
with respect
to y

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{\Delta y}$$



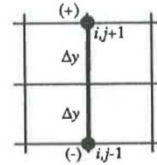
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Finite Difference Discretisation

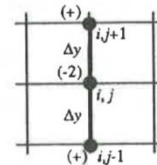
Second-order
central
difference
with respect
to y

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$



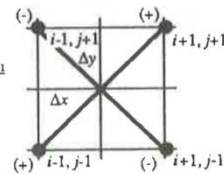
Second-order
central
second
difference
with respect
to y

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2}$$



Second-order
central
mixed
difference
with
respect
to x and y

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i-1,j+1} - u_{i+1,j-1}}{4\Delta x \Delta y}$$

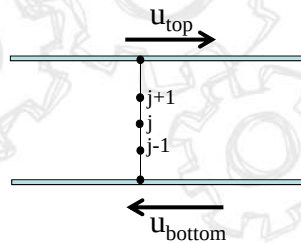


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Steady incompressible 1-D flow solution

Flow between parallel plates (neglecting gravitational body force)



$$\frac{\partial u}{\partial x} = 0 \quad (A)$$

$$\frac{\partial(\rho uu)}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \quad (B)$$

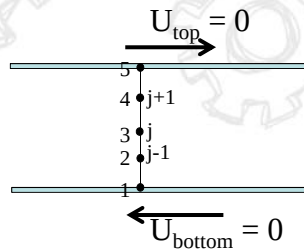
Combining (A) and (B) the resulting equation is

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \quad (C)$$

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Steady incompressible 1-D flow solution Finite-difference discretisation



The equivalent finite difference equation

$$\frac{\partial p}{\partial x} \frac{\Delta y^2}{\mu} = u_{j+1} - 2 u_j + u_{j-1} \quad (A)$$

$$\frac{\partial p}{\partial x} \frac{\Delta y^2}{\mu} = u_3 - 2 u_2 + u_1 \quad \text{Node 2}$$

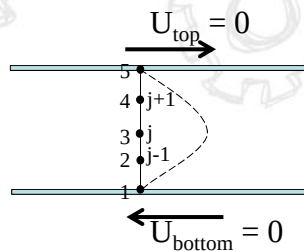
$$\frac{\partial p}{\partial x} \frac{\Delta y^2}{\mu} = u_4 - 2 u_3 + u_2 \quad \text{Node 3}$$

$$\frac{\partial p}{\partial x} \frac{\Delta y^2}{\mu} = u_5 - 2 u_4 + u_3 \quad \text{Node 4}$$



Steady incompressible 1-D flow solution Finite-difference discretisation

Given:
 $H = 0.1 \text{ m}$
 $\mu = 3.81 \text{ kg/m.s}$
 $\partial p / \partial x = -100 \text{ Pa/m}$



The solution is:

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.0246 \\ 0.0328 \\ 0.0246 \end{Bmatrix} \text{ m/s}$$

The equivalent finite difference equation

$$\frac{\partial p}{\partial x} \frac{\Delta y^2}{\mu} = u_{j+1} - 2 u_j + u_{j-1} \quad (A)$$

The system of equations is

$$-0.0164 = -2 u_2 + u_3 \quad \text{Node 2}$$

$$-0.0164 = u_4 - 2 u_3 + u_2 \quad \text{Node 3}$$

$$-0.0164 = u_3 - 2 u_4 \quad \text{Node 4}$$

In matrix form:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -0.0164 \\ -0.0164 \\ -0.0164 \end{Bmatrix}$$



Example of the MATLAB code for the flow between parallel plates

```
clear all;
clc;
clf;

H = 0.1; % height (m)
n = 8; % number of division
Utop = 0; % top plate velocity (m/s)
Ubottom = 0; % bottom plate velocity (m/s)
dpx = -100; % pressure gradient (Pa/m) (-) favorable (+) adverse
dyvisco = 3.81; % dynamic viscosity (kg/m.s)
npoint = n - 1; % number of unknown nodes
deltaY = H/n; % step size (m)
```

```
f(1:npoint) = dpx * deltaY^2 / dyvisco;
```

```
A(npoint,npoint) = zeros;
```

```
y(1) = 0;
```

```
y(n+1) = H;
```

```
for j = 2:npoint+1
    y(j) = y(j-1) + deltaY;
end
```

```
for j = 1:npoint-1
```

```
    A(j,j+1) = 1;
```

```
end
```

```
for j = 2:npoint
```

```
    A(j,j-1) = 1;
```

```
end
```

```
for j = 1:npoint
```

```
    A(j,j) = -2;
```

```
end
```

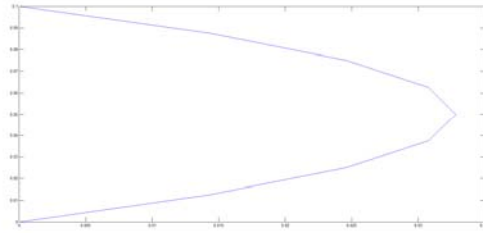
```
u(2:npoint+1) = A\F;
```

```
u(1) = Ubottom;
```

```
u(n+1) = Utop;
```

```
plot(u,y)
```

Note: This program is not generalized yet. You will need to improve / modify the code in exercise 2.



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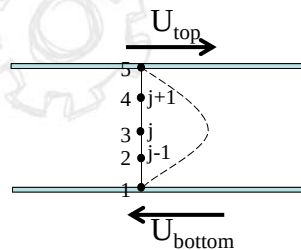
Exercises

1. Given:

case a : $H = 0.1$ m, $\mu = 3.81$ kg/m.s, $\partial p / \partial x = 0$, $U_{top} = 1$ m/s, $U_{top} = 0$ m/s

case b : $H = 0.1$ m, $\mu = 3.81$ kg/m.s, $\partial p / \partial x = -100$, $U_{top} = 1$ m/s, $U_{top} = 0$ m/s

case c : $H = 0.1$ m, $\mu = 3.81$ kg/m.s, $\partial p / \partial x = -100$, $U_{top} = 1$ m/s, $U_{top} = -1$ m/s



2. Develop a MATLAB code that plots the velocity profile and calculate the wall shear stress.

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