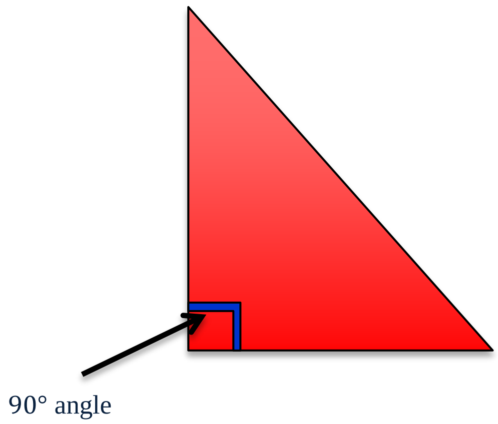
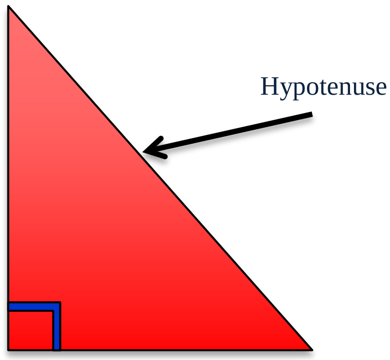
Pythagoras' Theorem

RIGHT-ANGLED TRIANGLES

You may remember that Pythagoras' Theorem has to do with right-angled triangles (unlike what a certain scarecrow in The Wizard of Oz tried to claim[1]). So, firstly, we will have a look at right-angled triangles and identify its important aspects.



We identify a right-angled triangle by the small right-angle mark at one vertex (corner) of the triangle. As shown in the diagram, the angle of that corner measures 90° which is a right angle. Because all the angles in a triangle add to 180°, the other two angels must add to 90°, so the right angle is the largest angle in the right-angled triangle. This also means that the side opposite the right angle (that is, the side that does not form one of the arms of the angle) is the largest in the triangle. It is given a special name: the *hypotenuse*.



We often use letters to identify the sides of a triangle, such as a, b, and c or other combinations. Here is an exercise on identifying the hypotenuse. You can check your answers with the solution at the end of the resource.

[1] In a famous scene the scarecrow proclaimed "*The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side*". Not only did he mix up right-angled and isosceles triangles, there are two other mistakes in the quote as well.

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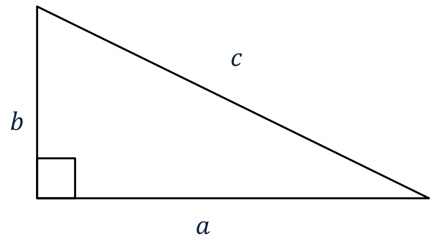
EXERCISE

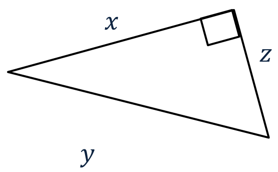
Identify the hypotenuse based on the lettering.

E.g.



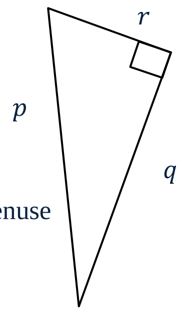
In this right-angled triangle, the hypotenuse is c.

1.



In this right-angled triangle, the hypotenuse is ________.

2.



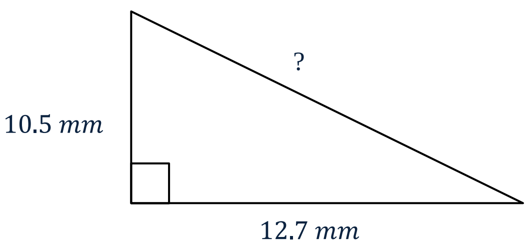
In this right-angled triangle, the hypotenuse is ________.

PYTHAGORAS' THEOREM

Pythagoras' Theorem has to do with the length of the hypotenuse and the other two sides in a right-angled triangle. (Although the theorem is identified by Pythagoras' name, it is doubtful that he actually was the first to prove it.) If the length of the hypotenuse in a right-angled triangle is c units and the other two sides are a units and b units long (as in the example triangle above), then it can be proved that[2]:

$$c^2 = a^2 + b^2.$$

So if we know the lengths of the other two sides of the triangle, we can calculate the length of the hypotenuse. Here is an example:



In this triangle, we are given two side lengths and the third is marked with a question mark. This third side is the hypotenuse, so we will use the letter c to represent its length in millimetres. For the other two side lengths it does not matter which we label a and which we label b, but you must make sure you have the hypotenuse correct[3].

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[2] Here we note the second mistake of the scarecrow; he talked about "square roots" while in fact the theorem is about squares.
[3] This was the third mistake of the scarecrow; he said we could use "any two sides" but we know that the remaining side must be the hypotenuse.

Using Pythagoras' Theorem we know that:

$$c^2 = a^2 + b^2,$$

So:

$$c^2 = 10.5^2 + 12.7^2,$$

So, using our calculator, we find that:

$$c^2 = 271.54.$$

But wait, our job is not done yet. We have found c^2, which is the square of the length of the triangle. We now need to "undo" the squaring by finding the square root of our answer. That means that:

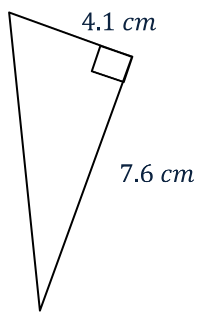
$$\begin{aligned} c &= \sqrt{271.54} \\ &= 16.4784708\ldots \end{aligned}$$

So, the length of the hypotenuse is approximately $16.5\ mm$.

Here is one for you to try. You can check your result with the solution at the end of this resource.

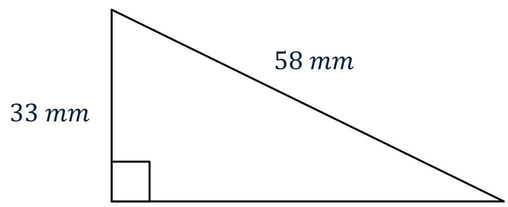
EXERCISES

3. Calculate the length of the hypotenuse of this right-angled triangle:



FINDING OTHER SIDES

What happens if we know the hypotenuse and want to find one other the other sides lengths? We can do this by rearranging our equation. Here is an example where we want to find the length of the side that is not marked:



This time we know the length of the hypotenuse, if we use our a, b, c labelling, then we can write $c = 58$. We also know one of the other side lengths so we can write $a = 33$ (we could also have chosen this one to be b) and so our unknown side length is b. We write down the formula:

$$c^2 = a^2 + b^2,$$

And replace the letters we know with their values:

$$58^2 = 33^2 + b^2.$$

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Using a calculator we obtain:

$$3364 = 1089 + b^2.$$

Now if we subtract 1089 from both sides we obtain:

$$2275 = b^2,$$

Which can also be written as:

$$b^2 = 2275.$$

Now we take the square root of 2275 to give us the value of the unknown side.

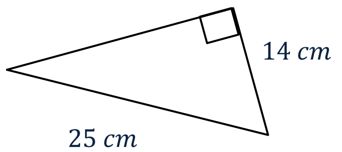
$$b = \sqrt{2275} = 47.69696007\ldots$$

So the length of the third side of our triangle is approximately $47.7\ mm$. (You should check with the other sides to ensure this makes sense, i.e. that it is less than the hypotenuse, but be careful as drawings may not be to scale!)

Here is one for you to try. You can check your result with the solution at the end of the resource.

EXERCISES

4.



PYTHAGOREAN TRIADS (OR TRIPLES)

You may have noticed that in the triangles we've looked at so far, even with those that have their first two side lengths give as whole numbers, the third side length has always come out as a long decimal (in fact they are all infinitely long with no repeating pattern. This is the case for most triangles; however some right-angled triangles do actually have three side lengths that are all whole numbers. Combinations of whole numbers that can form the side lengths of a right-angled triangle are called *Pythagorean Triads (or Triples).*

These sets are whole numbers that can be substituted into Pythagoras' formula and work. The most well-known example is (3,4,5):

We see that if our non-hypotenuse side lengths are 3 and 4 then Pythagoras' Theorem says that

$$\begin{aligned} c^2 &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25 \end{aligned}$$

And that this is exactly 5^2! So this is indeed a Pythagorean Triad.

Other examples are:

- (8,15,17)
- (5,12,13)
- (7,24,25)
- (6,8,10) – note that this is just twice our (3,4,5) triad!

In fact, if we multiply any triad by the same whole number, we will arrive at a new triad. For example (9,12,15) (which is three times (3,4,5)) or (16,30,34) (which is two times (8,15,17)). We have only listed a few of the more common examples, there are infinitely many non-similar (ones that aren't just a multiple of another) Pythagorean Triads.

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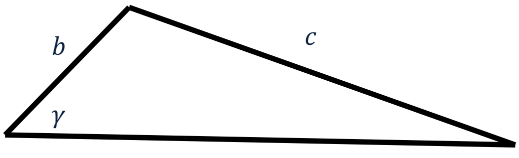
You might like to test this out in practice. Cut three pieces of string so that they measure $3\ m$, $4\ m$, and $5\ m$. Lay them out so that they form a right-angled triangle.

NON-RIGHT-ANGLED TRIANGLES – THE LAW OF COSINES

This is an extension section and is not required in order to use Pythagoras' Theorem. It also uses some trigonometry, if you want to read this section but are unfamiliar with trigonometry you might like to first read *Trigonometry*.

What about if a triangle isn't a right-angled triangle? Can we still find missing side lengths? The answer is yes! As long as we know one angle. Suppose that instead of being told our triangle has a right-angle we are told it has an angle γ (pronounced "gamma"). Then our triangle might look like this:



In this case, if we know a and b, we can calculate c by the *Law of Cosines* formula:

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

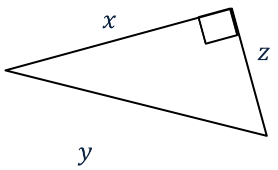
Notice that the formula starts off the same as the Pythagoras' Theorem formula, but contains an extra term to account for it not being a right-angled triangle. When $\gamma = 90°$ we can use that $\cos(90°) = 0$ to obtain Pythagoras' Theorem as a special case. It is also crucial to label to side opposite the known angle as c (just as it was previously crucial to make sure c was the hypotenuse).

We can also use the equation to find an unknown side length that is not opposite the angle. But this will require us solving a quadratic and we may end up with two, one, or even zero possible side lengths!

If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

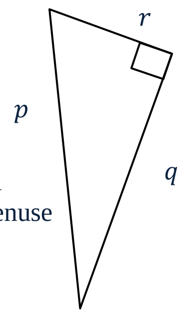
SOLUTIONS TO EXERCISES

1.



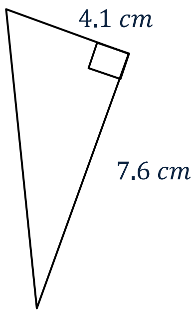
In this right-angled triangle, the hypotenuse is y.

2.



In this right-angled triangle, the hypotenuse is p.

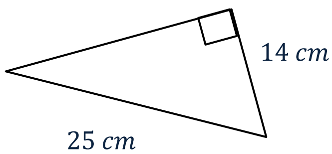
3. Calculate the length of the hypotenuse of this right-angled triangle:



$$c^2 = 4.1^2 + 7.6^2$$
$$= 74.57$$

So $c = \sqrt{74.57} \approx 8.64$, and the hypotenuse is approximately $8.6\ cm$ long.

4.



$$25^2 = 14^2 + b^2$$
$$625 = 196 + b^2$$
$$b^2 = 429$$

So $b = \sqrt{429} \approx 20.71$, so the unknown side length is approximately $20.7\ cm$ long.