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J.C.W. Rayner and D.J. Best

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National Institute for Applied Statistics Research Australia, University of Wollongong,
Wollongong NSW 2522. Phone +61 2 4221 5435, Fax +61 2 4221 4845.
Email: anica@uow.edu.au

SIMPLIFIED CALCULATION OF CMH STATISTICS

J.C.W. RAYNER^{1,2*} AND D.J. BEST²

University of Wollongong and University of Newcastle

Summary

The Cochran–Mantel–Haenszel tests are a suite of tests used to nonparametrically analyze tables of count data. Here the CMH mean scores and correlation test statistics are derived from first principles and are expressed in new forms that are easy to calculate give better insights for new users. The correlation statistic is shown to be proportional to the square of a linear combination of the Pearson correlations in each stratum. For the randomized block design the CMH mean scores statistic is shown to be a one to one function of the ANOVA F test statistic.

Keywords: Categorical data; contingency tables; cross-classified data; randomized blocks.

AMS SUBJECT CLASSIFICATION CODES: 62G10; 62F03

1. Introduction

The Cochran–Mantel–Haenszel (CMH) tests are a suite of nonparametric tests used to analyze tables $\{N_{ihj}\}$ of count data of a particular structure. Specifically N_{ihj} counts the number of times treatment i is classified into outcome category h in the j th stratum, $i = 1, \dots, t$, $h = 1, \dots, c$ and $j = 1, \dots, b$. Counts in different strata are independent, and the treatments present in each stratum are fixed by design. The CMH overall partial association (OPA), general association (GA), mean scores (MS) and correlation (C) tests

* Author to whom correspondence should be addressed.

¹ National Institute for Applied Statistics Research Australia, University of Wollongong, NSW 2522, Australia

Email: John.Rayner@newcastle.edu.au

² School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia

are described in, for example, Landis et al. (1978) and also Davis (2002). For these tests the null hypothesis of no association between the treatment and response variables in any of the b strata is tested against increasingly restrictive alternatives.

Software that applies the CMH methodology often requires data entry in strata, and if there are many strata this can be tedious. Where possible we seek methods that reduce this tedium. For the particular case of the non-ordinal CMH tests, Best and Rayner (2017) point out that very simple analogues of the CMH OPA and GA statistics can be calculated from Pearson statistics. These have asymptotic distributions the same as the corresponding CMH statistics and there exists standard easily used software that may be used to calculate the statistics.

Here, when the design is randomised blocks, we show the CMH MS test statistic is a simple function of the ANOVA F test statistic that is very frequently available using click and point software. This makes its calculation both quick and convenient.

For the CMH C test a new form of the test statistic that involves sums of squares and sums of products of the responses and treatments is derived. Attention is drawn to some particular cases. The CMH C test statistic is simply related to the Pearson correlation coefficients of individual strata.

In order to access these results in Sections 2 and 3 the CMH MS and C test statistics are defined with a little more mathematical detail than is usual in most presentations. Section 4 gives some examples, with a focus on convenient calculation to enable new users to develop intuition and avoid, initially at least, a ‘black box’ mentality. Section 5 focuses on the randomised block design.

2. Derivation of the CMH mean scores test statistic

As in the previous section suppose N_{ihj} counts the number of times treatment i is classified into outcome category h in the j th stratum, $i = 1, \dots, t$, $h = 1, \dots, c$ and $j = 1, \dots, b$. A score b_{hj} is assigned to the h th response on the j th stratum. The score sum for treatment i in stratum j is $\sum_{h=1}^c b_{hj}N_{ihj} = M_{ij}$. The traditional derivation of the CMH tests requires conditional inference and distribution theory assuming a product over strata of extended hypergeometric distributions. Using this product distribution M_{ij} has expectation $E[M_{ij}] = n_{i\cdot j} \sum_{h=1}^c b_{hj}n_{\cdot hj} / n_{\cdot\cdot j}$ under the null hypothesis of no mean effects. See Landis et al. (1978) for details.

If $M_j = (M_{ij})$ then inference about the population mean counts are based on $M = \sum_{j=1}^b M_j$ through the quadratic form

$$S_M = (M - E[M])^T \text{cov}^{-1}(M)(M - E[M])$$

in which all unknown parameters are estimated by maximum likelihood under the null hypothesis. The asymptotic distribution of S_M is χ_{t-1}^2 . Since $\text{cov}(M)$ is not of full rank a generalised inverse such as the Moore-Penrose inverse can be used.

To specify the CMH MS test statistic note that $E[M] = \sum_{j=1}^b E[M_j]$ and as strata are independent $\text{cov}(M) = \sum_{j=1}^b \text{cov}(M_j)$. The derivation of $\text{cov}(M_j)$ requires routine but tedious algebra. If δ_{uv} is the Kronecker delta, = 1 if $u = v$ and zero otherwise, then using standard distribution theory for the product extended hypergeometric distribution, the covariance between N_{ihj} and $N_{i'h'j}$ is

$$n_{i\bullet j} n_{\bullet h'j} (\delta_{ii'} n_{\bullet\bullet j} - n_{i'\bullet j}) (\delta_{hh'} n_{\bullet\bullet j} - n_{\bullet h'j}) / \{n_{\bullet\bullet j}^2 (n_{\bullet\bullet j} - 1)\}.$$

It follows that

$$\text{var}(\sum_{h=1}^c b_{hj} N_{ihj}) = n_{i\bullet j} (n_{\bullet\bullet j} - n_{i\bullet j}) S_j^2 / \{n_{\bullet\bullet j}^2 (n_{\bullet\bullet j} - 1)\}$$

and

$$\text{cov}(\sum_{h=1}^c b_{hj} N_{ihj}, \sum_{h'=1}^c b_{h'j} N_{i'h'j}) = -n_{i\bullet j} n_{i'\bullet j} S_j^2 / \{n_{\bullet\bullet j}^2 (n_{\bullet\bullet j} - 1)\}.$$

Now define $p_{i\bullet j} = n_{i\bullet j} / n_{\bullet\bullet j}$ and

$$S_j^2 = \frac{n_{\bullet\bullet j} \sum_{h=1}^c b_{hj}^2 n_{\bullet h'j} - (\sum_{h=1}^c b_{hj} n_{\bullet h'j})^2}{(n_{\bullet\bullet j} - 1)}.$$

Then $\text{cov}(M_j) = S_j^2 \{\text{diag}(p_{i\bullet j}) - (p_{i\bullet j} p_{i'\bullet j})\}$. Now put

$$D_j = \text{diag}(\sqrt{p_{i\bullet j}}) \text{ and } u_j = (\sqrt{p_{i\bullet j}}).$$

Then $\text{cov}(M_j) = S_j^2 D_j \{I_t - u_j u_j^T\} D_j$. If $V = (M_{i\bullet} - \sum_{j=1}^b n_{i\bullet j} \sum_{h=1}^c b_{hj} n_{\bullet hj} / n_{\bullet\bullet j})$ it follows that if $+$ denotes the Moore-Penrose inverse then S_M can be calculated from

$$V^T [\sum_{j=1}^b S_j^2 D_j \{I_t - u_j u_j^T\} D_j]^+ V.$$

3. Derivation of the CMH Correlation test statistic

In this section we continue to refer to responses, treatments and strata, although the choice of which variable is treatment and which is response is arbitrary.

To define the CMH C test statistic first assign on the j th stratum the scores a_{ij} to the i th treatment and b_{hj} to the h th response and then put

$$C_j = \sum_i \sum_h a_{hi} b_{hj} \{N_{ihj} - E[N_{ihj}]\} \text{ and } C = \sum_j C_j.$$

The CMH C test statistic is $C^2/\text{var}(C) = S_C$ say. The derivation of $\text{var}(C)$ is routine using Kronecker products, but less so otherwise. First define

$$a_j = (a_{1j}, \dots, a_{tj})^T, b_j = (b_{1j}, \dots, b_{cj})^T, \text{ and } N_j = (N_{11j}, \dots, N_{1cj}, \dots, N_{t1j}, \dots, N_{tcj})^T.$$

Then $C_j = (a_j \otimes b_j)^T (N_j - E[N_j])$ and

$$\begin{aligned} \text{var}(C_j) &= E[(a_j \otimes b_j)^T (N_j - E[N_j]) (N_j - E[N_j])^T (a_j \otimes b_j)] = \\ &= (a_j \otimes b_j)^T E[(N_j - E[N_j]) (N_j - E[N_j])^T] (a_j \otimes b_j) = \\ &= (a_j \otimes b_j)^T \text{cov}(N_j) (a_j \otimes b_j). \end{aligned}$$

Now write

$$p_{\bullet hj} = n_{\bullet hj} / n_{\bullet\bullet\bullet}, \quad V_{Tj} = \text{diag}(p_{i\bullet j}) - (p_{i\bullet j})(p_{i\bullet j})^T \quad \text{and} \quad V_{Cj} = \text{diag}(p_{\bullet hj}) - (p_{\bullet hj})(p_{\bullet hj})^T.$$

From Landis et al. (1978)

$$\text{cov}(N_j) = \frac{n_{\bullet\bullet j}^2}{n_{\bullet\bullet j} - 1} (V_{Tj} \otimes V_{Cj}).$$

Hence

$$\text{var}(C_j) = \frac{n_{\bullet\bullet j}^2}{n_{\bullet\bullet j} - 1} (a_j^T V_{Tj} a_j \otimes b_j^T V_{Cj} b_j) = \frac{n_{\bullet\bullet j}^2}{n_{\bullet\bullet j} - 1} (a_j^T V_{Tj} a_j)(b_j^T V_{Cj} b_j)$$

because both factors in the Kronecker product are scalars. Finally $\text{var}(C) = \sum_j \text{cov}(C_j)$

because strata are mutually independent. The Central Limit Theorem assures the asymptotic normality of C , so as the total sample size $n_{\bullet\bullet\bullet} = n_{\bullet\bullet 1} + \dots + n_{\bullet\bullet s}$ approaches infinity S_C has asymptotic distribution χ_1^2 . The CMH correlation statistic S_C is now fully, if not conveniently, specified. We now progress towards doing so. First

$$E[N_{ihj}] = n_{i\bullet j} n_{\bullet hj} / n_{\bullet\bullet j} \quad \text{and}$$

$$C_j = \sum_i \sum_h a_{ij} b_{hj} \{N_{ihj} - E[N_{ihj}]\} = \sum_i \sum_h a_{ij} b_{hj} N_{ihj} - \{\sum_i a_{ij} n_{i\bullet j}\} \{\sum_h b_{hj} n_{\bullet hj}\} / n_{\bullet\bullet j}.$$

Using the definitions of V_{Tj} and V_{Cj} we have

$$n_{\bullet\bullet j} a_j^T V_{Tj} a_j = \sum_i a_{ij}^2 n_{i\bullet j} - (\sum_i a_{ij} n_{i\bullet j})^2 / n_{\bullet\bullet j} \quad \text{and}$$

$$n_{\bullet\bullet j} b_j^T V_{Cj} b_j = \sum_h b_{hj}^2 n_{\bullet hj} - (\sum_h b_{hj} n_{\bullet hj})^2 / n_{\bullet\bullet j}.$$

On stratum j now define

$$S_{XXj} = \sum_i a_{ij}^2 n_{i\cdot j} - (\sum_i a_{ij} n_{i\cdot j})^2 / n_{\cdot\cdot j},$$

$$S_{XYj} = \sum_i \sum_h a_{ij} b_{hj} N_{ihj} - \{ \sum_i a_{ij} n_{i\cdot j} \} \{ \sum_h b_{hj} n_{\cdot hj} \} / n_{\cdot\cdot j} \text{ and}$$

$$S_{YYj} = \sum_h b_{hj}^2 n_{\cdot hj} - (\sum_h b_{hj} n_{\cdot hj})^2 / n_{\cdot\cdot j}.$$

With appropriate divisors the S_{XXj} etc. give unbiased estimators of the stratum variances and covariances. With these definitions

$$C_j = S_{XYj}, \quad n_{\cdot\cdot j} a_j^T V_{Tj} a_j = S_{XXj} \text{ and } n_{\cdot\cdot j} b_j^T V_{Cj} b_j = S_{YYj}.$$

Finally since $C = \sum_j C_j$ and $\text{var}(C) = \sum_j \text{cov}(C_j)$

$$S_C = C^2 / \text{var}(C) = \frac{\{ \sum_j S_{XYj} \}^2}{\sum_j \{ S_{XXj}^2 S_{YYj}^2 / (n_{\cdot\cdot j} - 1) \}}.$$

New users will gain few insights from the usual expression of S_C . The expression here uses the S_{XYj} etc., familiar in formulae for regression coefficients. Intuition may be developed by considering three special cases:

- (1) the data consists of one stratum only;
 - (2) the treatment scores are independent of the strata: $a_{ij} = a_i$ for all i and j ; and
 - (3) the randomised block design.
- In the second case S_{XX} is constant over strata. This gives a slight simplification of the S_C formula. See the *Jams Example* in Section 4.
 - If the data come from a randomised block design a considerable simplification is possible if the same treatment and response scores are used on each stratum or block. See section 5.
 - In the first case the CMH correlation statistic simplifies to

$$S_C = \frac{(n_1 - 1) S_{XY1}^2}{S_{XX1}^2 S_{YY1}^2} = (n_1 - 1) r_p^2,$$

in which r_p is the Pearson correlation coefficient. This is well known. See, for example, Davis (2002, p.253).

If we now write r_{pj} for the Pearson correlation in the j th stratum, it follows that since $S_{xyj} = r_{pj}S_{xxj}S_{yyj}$

$$S_C = \frac{\{\sum_j r_{pj}S_{xxj}S_{yyj}\}^2}{\sum_j \{S_{xxj}^2 S_{yyj}^2 / (n_{\bullet\bullet j} - 1)\}}$$

S_C is proportional to the square of a linear combination of the Pearson correlations in each stratum. The proportionality factor ensures S_C has the χ_1^2 distribution. This formula demonstrates how the Pearson correlations in each stratum contribute to the overall correlation measure.

4. Examples

In this section three examples are given demonstrating the use of the formulae for S_M and S_C and the simplifications mentioned above. There is, of course, software available for calculating the CMH statistics. However using the simplifications given above it is not difficult to calculate CMH statistics without resorting to software with which some users are unfamiliar and for which data entry is sometimes tedious.

The first example, the *Homosexual Marriage Example*, is the most general. In the second, the *Whiskey Example*, there is just one stratum. In the *Jams Example* although there are eight strata hand calculation is possible if a little tedious. Because S_{XX} is constant over strata it is not too much extra work to calculate S_{XY} , S_{YY} , the Pearson correlation, the CMH correlation statistic and its p-value on each stratum.

Homosexual Marriage Example. See the data given in Table 1. After assigning scores of 1, 2 and 3 for the responses agree, neutral and disagree respectively to the proposition “Homosexuals should be able to marry” and 1, 2 and 3 for the religions fundamentalist, moderate and liberal respectively, Agresti (2002) finds the CMH GA test statistic takes the value 19.76 with χ_4^2 p-value 0.0006, the CMH MS test statistic takes the value 17.94

with χ^2_2 p-value 0.0001, while the CMH C test statistic takes the value 16.83 with χ^2_1 p-value less than 0.0001. Using these tests we conclude there is strong evidence of differences in religion and of an association between the proposition responses and religion.

Table 1. Opinions on homosexual marriage by religious beliefs and education levels for ages 18 to 25

		Homosexuals should be able to marry		
Education	Religion	Agree	Neutral	Disagree
School	Fundamentalist	6	2	10
	Moderate	8	3	9
	Liberal	11	5	6
College	Fundamentalist	4	2	11
	Moderate	21	3	5
	Liberal	22	4	1

Noting that stratum 1 is school and stratum 2 is college we find $S_{XX1} = 50$, $S_{XY1} = -9$ and $S_{YY1} = 39.7333$, $r_{P1} = -0.2019$ and the CMH C statistic for school takes the value 2.4055 with χ^2_1 p-value 0.1209. Similarly $S_{XX2} = 51.6712$, $S_{XY2} = -23.8904$, $S_{YY2} = 42.6301$, $r_{P2} = -0.5090$ and the CMH C statistic for college takes the value 18.6558 with χ^2_1 p-value 0.0000. From these the value of the CMH C statistic and its χ^2 p-value are confirmed: we found S_C takes the value 16.8328 with χ^2_1 p-value 0.0000. Clearly there is an insignificant Pearson correlation for schools and a highly significant Pearson correlation for college. The latter dominates the former, so that overall there is strong evidence of a correlation effect: as religion becomes increasingly liberal there is greater agreement with the proposition that homosexuals should be able to marry. This is due mainly to the stratum college.

Whiskey Example. O'Mahony (1986, p.363) gave the data in Table 2, which are analysed in Rayner and Best (2001). They use mid-rank scores and find the Spearman correlation,

which takes the value 0.73. In testing if this is zero against two-sided alternatives they give a Monte Carlo p-value of 0.09 and an asymptotic p-value of 0.04.

Table 2. Cross classification of age and whiskey

Years of maturing	Grade			Total
	First	Second	Third	
One	0	0	2	2
Five	1	1	1	3
Seven	2	1	0	3
Total	3	2	3	8

Using scores 1, 2 and 3 for grade and 1, 5 and 7 for years of maturity, we find $S_{XX} = 6$, $S_{XY} = -12$ and $S_{YY} = 43.5$. It follows that the Pearson correlation is -0.7428 and the CMH C statistic takes the value 3.8621 with χ_1^2 p-value 0.0494. There is some evidence that as maturity increases so does the grade of the whiskey.

Jams Example. Although the data satisfy as a randomised block design in the first instance we only make use of the fact that S_{XX} is constant over strata. This example is revisited in the next section.

Table 3. JAR codes for sweetness of three plum jams

Judge	Jam		
	A	B	C
1	3	2	3
2	4	5	4
3	3	2	3
4	1	4	2
5	2	4	2
6	1	3	3
7	2	5	4
8	2	5	2

Table 4. Analysis of Jams data

	Stratum							
	1	2	3	4	5	6	7	8
S_{XY}	0	0	0	1	0	2	2	0
S_{YY}	0.6667	0.6667	0.6667	4.6667	2.6667	2.6667	4.6667	6
r_P	0	0	0	0.3273	0	0.8660	0.6547	0
S_C	0	0	0	0.2143	0	1.5	0.8571	0
p-value	1	1	1	0.6434	1	0.2207	0.3545	1

Three plum jams, A, B and C are given just about right (JAR) sweetness codes by eight judges according to Table 3. Here 1 denotes *not sweet enough*, 2 *not quite sweet enough*, 3 *just about right*, 4 *a little too sweet* and 5 *too sweet*. We find $S_{XX} = 2$ on all strata and $S_C = 1.1029$ with χ_1^2 p-value 0.2936. There is no significant correlation effect, which would, if present, indicate that as we pass from jam A to B to C there is increasing or decreasing sweetness. It could be that overall there is no correlation effect with the contrary being the case in a minority of strata. That is not the case here; no stratum shows any evidence of a slight correlation effect. Here this is hardly surprising; there are only three treatments in each stratum.

If there is interest in the contributions to the correlation from individual strata this is a reasonable approach. However, if there is not, then for the randomised blocks design with the same treatment scores in each stratum a considerable simplification is possible. This is considered in the next section.

5. The randomised blocks design

Arguably the CMH methodology has been developed to be consistent with the randomised blocks design. If there were two factors of interest the second factor could replace the strata, but these would no longer be independent, and a different model would be needed. As developed the CMH methodology doesn't apply to Latin square, multifactor and many other designs. However it is an extremely important analysis tool

for randomised blocks when the responses are categorical rather than continuous. Davis (2002, Chapter 8) uses CMH methods to analyse repeated measures categorical data.

5.1 The CMH MS statistic for the randomised block design

In the case of the randomised block design a considerable simplification of the CMH MS statistic is possible. In this case $n_{i\bullet j} = 1$ for all i and j . Consequently $n_{\bullet\bullet j} = t$, $n_{i\bullet\bullet} = b$, $n_{\bullet\bullet\bullet} = bt$ and $p_{i\bullet j} = n_{i\bullet\bullet} / n_{\bullet\bullet\bullet} = 1/t$. Substituting in the definitions given in Section 2 gives $D_j = I_t / \sqrt{t}$, $u_j = 1_t / \sqrt{t} = u$ say, and $(t-1)S_j^2 = t \sum_h b_{hj}^2 n_{\bullet hj} - (\sum_h b_{hj} n_{\bullet hj})^2$. Thus

$$\text{cov}(M_j) = S_j^2 D_j \{I_t - 1_t 1_t^T / t\} D_j = S_j^2 \{I_t - 1_t 1_t^T / t\} / t \text{ and}$$

$$\text{cov}(M) = \sum_{j=1}^b \text{cov}(M_j) = \{I_t - 1_t 1_t^T / t\} \sum_j S_j^2 / t.$$

Now $1_t 1_t^T / t$ is idempotent, as is $I_t - 1_t 1_t^T / t$, and as any idempotent matrix is its own Moore-Penrose inverse (denoted here by '+'),

$$\text{cov}^+(M) = t \{I_t - 1_t 1_t^T / t\} / (\sum_j S_j^2).$$

In the definition of S_M ,

$$M - E[M] = (\sum_j M_{ij}) - (\sum_{h,j} b_{hj} n_{\bullet hj}) / t.$$

Put $V = (V_i) = M - E[M]$ so that $V_i = M_{i\bullet} - \sum_{h,j} b_{hj} n_{\bullet hj} / t$. The V_i are differences between the sum of the treatment scores over all strata and the mean of these. Now

$$1_t^T V = \sum_i (M_{ij} - E[M_{ij}]) = \sum_{h=1}^c b_{hj} n_{\bullet hj} - n_{\bullet\bullet j} \sum_{h=1}^c b_{hj} n_{\bullet hj} / n_{\bullet\bullet j} = 0.$$

It follows that

$$S_M = V^T \text{cov}^+(M)V = t\{\sum_i V_i^2\}/\{\sum_j S_j^2\}.$$

To simplify this first put $x_{ij} = \sum_h b_{hj} N_{ihj}$. Note that $x_{\bullet j} = \sum_h b_{hj} n_{\bullet hj}$. On the j th block there is only one observation of treatment i , so for only one value of h is $n_{ihj} = 1$; otherwise $n_{ihj} = 0$. Thus, for example, $x_{ij}^2 = \sum_h b_{hj}^2 N_{ihj}$. Consider the data set $\{x_{ij}\}$. For the two-way ANOVA of these data write TSS for the treatments sum of squares and ESS for the error sum of squares.

If the data $\{x_{ij}\}$ are analysed as a one-way ANOVA or completely randomised design with blocks as treatments then the error sum of squares is $TSS + ESS$ and is given by

$$\begin{aligned} TSS + ESS &= \sum_{i,j} (x_{ij} - x_{\bullet j}/t)^2 = \sum_{i,j} x_{ij}^2 - \sum_j x_{\bullet j}^2/t = \sum_j \{\sum_i x_{ij}^2 - x_{\bullet j}^2/t\} = \\ &= \sum_j \{\sum_i (\sum_h b_{hj}^2 N_{ihj}) - (\sum_h b_{hj} n_{\bullet hj})^2/t\} = \sum_j \{\sum_h b_{hj}^2 n_{\bullet hj} - (\sum_h b_{hj} n_{\bullet hj})^2/t\} = (t-1) \sum_j S_j^2/t. \end{aligned}$$

Further, for the two-way ANOVA the treatment sum of squares is (note that $\bar{x}_{i\bullet}$ means the mean of $\{x_i\}$ while x_{\bullet} means the sum of the $\{x_i\}$)

$$\begin{aligned} TSS &= \sum_{i,j} (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2 = b \sum_i \{x_{i\bullet}/b - x_{\bullet\bullet}/(bt)\}^2 \\ &= \frac{1}{b} \sum_i \{\sum_{h,j} b_{hj} N_{ihj} - \sum_{h,j} b_{hj} n_{\bullet hj}/t\}^2 = \frac{1}{b} \sum_i V_i^2. \end{aligned}$$

It follows that

$$S_M = t\{\sum_i V_i^2\}/\{\sum_j S_j^2\} = \frac{bt TSS}{t\{TSS + ESS\}/(t-1)} = \frac{b(t-1) F}{(b-1 + F)}$$

on using $TMS = TSS/(t - 1)$, $EMS = ESS/\{(b - 1)(t - 1)\}$ and $F = TMS/EMS$, the F test statistic for treatments in the two-way ANOVA. Coincidentally this is the same relationship relating the Friedman test statistic and the F test statistic using the ranks as data.

The significance of this result is twofold. First, calculations can be done using standard ANOVA routines, such as those available from click and point software. Second, p-values can be found using either S_M and its χ^2_{t-1} distribution or F and its $F_{(t-1),(b-1)(t-1)}$ distribution. An empirical study would be required to assess which is the more reliable in the sense of closeness to, say, permutation test p-values. In the examples we have analysed there has been little difference in these methods.

5.2 The CMH C statistic for the randomised block design

Suppose, as is almost always the case, that the same treatment and response scores are used on each stratum or block. Then $a_{ij} = a_i$ say, for all i and j , and $b_{hj} = b_h$ say, for all h and j . As $n_{i\bullet j} = 1$ for all i and j , and as a consequence $n_{\bullet\bullet j} = t$ for all j $S_{XXj} = \sum_i a_{ij}^2 n_{i\bullet j} - (\sum_i a_{ij} n_{i\bullet j})^2 / n_{\bullet\bullet j} = \sum_i a_i^2 - (\sum_i a_i)^2 / t$, which is independent of j , so we may write $S_{XXj} = S_{XX}$. It is *not* true that S_{YYj} is independent of j .

Next we focus on S_{YYj} . Put $x_{ij} = \sum_h b_{hj} N_{ihj}$. As in section 5.1 $x_{ij}^2 = \sum_h b_{hj}^2 N_{ihj}$. Consider the data set $\{x_{ij}\}$. If these are analysed as a one-way ANOVA or completely randomised design with blocks as treatments then the error sum of squares is

$$\begin{aligned} \sum_{i,j} (x_{ij} - \bar{x}_{\bullet j})^2 &= \sum_{i,j} x_{ij}^2 - t \sum_j \bar{x}_{\bullet j}^2 = \sum_j \{ \sum_i x_{ij}^2 - x_{\bullet j}^2 / t \} = \\ \sum_j \{ \sum_i (\sum_h b_{hj}^2 N_{ihj}) - (\sum_h b_{hj} n_{\bullet hj})^2 / t \} &= \sum_j \{ \sum_h b_{hj}^2 n_{\bullet hj} - (\sum_h b_{hj} n_{\bullet hj})^2 / t \} = \sum_j S_j^2. \end{aligned}$$

Finally we focus on S_{XYj} . Write $\bar{a} = \sum_i a_i / t$. Then

$$\sum_j C_j = \sum_{i,h,j} a_{ij} b_{hj} \{N_{ihj} - E[N_{ihj}]\} = \sum_{i,h,j} a_i b_{hj} \{N_{ihj} - \frac{n_{\bullet hj}}{t}\}.$$

Now

$$\sum_{i,h,j} \{a_i - \bar{a}\} b_{hj} \{N_{ihj} - \frac{n_{\bullet hj}}{t}\} = \sum_{i,h,j} \{a_i - \bar{a}\} b_h N_{ihj} - \sum_{i,h,j} \{a_i - \bar{a}\} b_{hj} n_{\bullet hj} / t.$$

Since

$$\sum_{i,h,j} \{a_i - \bar{a}\} b_{hj} n_{\bullet hj} / t = \sum_i \{a_i - \bar{a}\} \sum_h b_h \sum_j n_{\bullet hj} / t = \{\sum_h n_{\bullet h} b_h\} \sum_i \{a_i - \bar{a}\} / t = 0$$

$$C = \sum_j C_j = \sum_i \{a_i - \bar{a}\} \sum_{h,j} b_h N_{ihj}.$$

Note that $\sum_{h,j} b_h N_{ihj}$ is the sum of the i th treatment scores over responses and blocks, and these are usually easy to calculate directly or are readily available in most packaged analyses. The $\{a_i - \bar{a}\}$ are the centred treatment scores. Thus

$$S_C = \frac{(t-1)C^2}{S_{XX}^2 \sum_j S_{YYj}^2}$$

in which C is the sum of the products of the treatment sums and the centred treatment scores, S_{XX}^2 is the sum of the squares of the centred treatment scores and $\sum_j S_{YYj}^2$ may be read from the output for a one-way ANOVA.

Jams Example continued. For the jams data $a_i = 1, 2$ and 3 , so $S_{XX}^2 = 2$. From a one-way ANOVA with judges as treatments, $\sum_j S_{YYj}^2 = 22 \frac{2}{3}$. This can be confirmed by summing across the S_{YY} row in Table 4. By summing the columns in Table 3 the treatment score sums are found to be 18, 30 and 23, and since the centred treatment scores are $-1, 0$ and 1 , $C = -18 + 0 + 23 = 5$. Substituting gives $S_C = 2 \cdot 25 / \{2 \cdot 68 / 3\} = 75 / 68 = 1.1029$ with χ_1^2 p-value 0.2936, as previously.

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