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Abstract
When ties occur in the Kruskal-Wallis, Friedman and Durbin tests it is usual to use mid-ranks to calculate the test statistics. This, however, inflates the test p-values. Corrections are needed. One such involves the sum of the squares of the mid-ranks. For hand calculation this is more cumbersome than is necessary. Simpler formulae are given here. These use the number of ranks in groups of ties. Both approaches agree and reduce to the uncorrected formulae when there are no ties.

KEYWORDS*:* Balanced incomplete block design; completely randomised design; jams example; randomised block design

AMS SUBJECT CLASSIFICATION CODES: 62G; 62F

1. Introduction

When ties occur in the Kruskal-Wallis, Friedman and Durbin tests it is usual to use mid-ranks to calculate the test statistics. This, however, inflates the test p-values. A correction is needed. One approach involves the sum of the squares of the mid-ranks. For hand calculation this is more cumbersome than is necessary. Simpler formulae are given here. These use the number of ranks in groups of ties. Both approaches agree and reduce to the uncorrected formulae when there are no ties.

The following sections give the Kruskal-Wallis, Friedman and Durbin tests, then the form of the adjusted test statistics using the sum of the squares of the mid-ranks, followed by the adjusted test statistics using the number of ranks in groups of ties. An example is given.

2. The Kruskal-Wallis, Friedman and Durbin Tests for untied data

Suppose we have distinct (untied) observations, with y_{ij} being the jth of n_i observations on the ith of t treatments. The model assumed is the completely randomized design, sometimes called the one-way layout and sometimes the one-way analysis of variance. All $n_1 + \ldots + n_t = n$ observations are combined, ordered and ranked. For $i = 1, \ldots, t$ the sums of the ranks for treatment i, R_i, is calculated. The Kruskal-Wallis test statistic KW is given by

$$KW = \frac{12}{n(n+1)} \sum_{i=1}^{t} \frac{R_i^2}{n_i} - 3(n+1).$$

For the randomised block design suppose we have distinct (untied) observations, y_{ij}, this being the ith of t treatments on the jth of b blocks. The observations are ranked within each block and R_i, the sum of the ranks for treatment i over all blocks is calculated for $i = 1, \ldots, t$. The Friedman test statistic is

$$S = \frac{12}{bt(t+1)} \sum_{i=1}^{t} R_i^2 - 3b(t+1).$$

In the balanced incomplete block design each of the b blocks contains k experimental units, each of the t treatments appears in r blocks, and every treatment appears with every other treatment precisely λ times. Necessarily

$$k < t,\ r < b,\ bk = rt, \text{ and } \lambda(t-1) = r(k-1).$$

Treatments are ranked within each block. Durbin's statistic, D, is given by

$$D = \frac{12(t-1)}{bk(k^2-1)} \sum_{i=1}^{t} R_i^2 - \frac{3r(t-1)(k+1)}{(k-1)}$$

in which R_i is the sum of the ranks given to treatment i, $i = 1, \ldots, t$.

3. Adjusting for ties for the Kruskal-Wallis, Friedman and Durbin Tests using the sum of squares of the ranks

If ties occur mid-ranks are used and corrections to the test statistics are given, for example, in Thas et al. (2012). These corrections are based on the sum of squares of all of the mid-ranks. We will need the identities

$$1 + \ldots + p = p(p+1)/2,$$
$$1^2 + \ldots + p^2 = p(p+1)(2p+1)/6$$

$$p(p + 1)(2p+1)/6 - \{ p(p + 1)/2\}^2 = (p - 1)p(p + 1)/12.$$

If ties occur and the Kruskal-Wallis test statistic uses mid-ranks, one adjustment to the test statistic is

$$KW_A = \frac{(n-1)\{\sum_i \frac{R_i^2}{n_i} - \frac{n(n+1)^2}{4}\}}{\sum_{i,j} r_{ij}^2 - \frac{n(n+1)^2}{4}}$$

in which r_{ij} is the overall rank of y_{ij}. If there are no ties then these ranks are 1, 2, …, n, and the denominator in KW_A, by the third of the given identities is $(n - 1)n(n + 1)/12$. With this simplification KW_A reduces to KW.

If ties occur and the Friedman test statistic uses mid-ranks, then again, the test statistic needs adjusting. One such adjustment is

$$S_A = \frac{(t-1)\{\sum_i R_i^2 - \frac{b^2t(t+1)^2}{4}\}}{\sum_{i,j} r_{ij}^2 - \frac{bt(t+1)^2}{4}}$$

in which r_{ij} is the rank of treatment i on block j. If there are no ties then on block j the ranks are 1, 2, …, t. Again using the given identities $\sum_{i,j} r_{ij}^2 = bt(t + 1)(2t+1)/6$ and the denominator in S_A is $b(t - 1)t(t + 1)/12$. With this simplification S_A reduces to S.

If ties occur and the Durbin test statistic uses mid-ranks, then an adjusted test statistic is given by

$$D_A = \frac{(t-1)\{\sum_i R_i^2 - \frac{rbk(k+1)^2}{4}\}}{\sum_{i,j} r_{ij}^2 - \frac{bk(k+1)^2}{4}}$$

in which r_{ij} is the rank of treatment i on block j. If there are no ties then on block j the ranks are 1, 2, …, k. Using the given identities $\sum_{i,j} r_{ij}^2 = bk(k + 1)(2k+1)/6$ and the denominator in S_A is $b(k - 1)k(k + 1)/12$. With this simplification D_A reduces to D.

Alternative expressions are available for the adjusted test statistics. These expressions use the number of tied data in a group of mid-ranks and is easier to calculate by hand, especially when there are few ties.

4. Adjusting for ties for the Kruskal-Wallis, Friedman and Durbin Tests using the number of tied data in a group of mid-ranks

Common to all three test statistics is the following approach. Consider a group of three tied ranks. Instead of the ranks being h, $h + 1$ and $h + 2$, using mid-ranks they are $h + 1$,

h + 1 and h + 1. This causes no change in the rank sum but the change in the rank sum of squares is $h^2 + (h + 1)^2 + (h + 2)^2 – 3(h + 1)^2 = 2$. When looking at the changes for, say, the gth group which is of size t_g, we find the difference is well-modelled by $(t_g^3 - t_g)/12$ for t_g = 2, 3, …. The result can be proven by induction. It follows that when passing from untied to tied data using mid-ranks for p observations, the sum of squares is reduced by the aggregation of the corrections $(t_g^3 - t_g)/12$ for all groups of tied observations. Thus $\{1^2 + … + p^2\}$ becomes $\{1^2 + … + p^2\} – \sum_g (t_g^3 - t_g)/12$.

For the Kruskal-Wallis test statistic the denominator becomes

$$\{1^2 + … + n^2\} – \sum_g (t_g^3 - t_g)/12 – \{n(n + 1)/2\}^2$$
$$= (n – 1)n(n + 1)/12 – \sum_g (t_g^3 - t_g)/12$$
$$= C(n – 1)n(n + 1)/12$$

in which

$$C = 1 – \sum_g (t_g^3 - t_g)/\{(n-1)n(n+1)\}.$$

Thus $KW_A = KW/C$.

For the Friedman test statistic suppose that on the jth block the gth group is of size $t_{g,j}$. The denominator becomes

$$b\{1^2 + … + t^2\} – \sum_{g,j} (t_{g,j}^3 - t_{g,j})/12 – b\{t(t + 1)/2\}^2$$
$$= b(t – 1)t(t + 1)/12 – \sum_g (t_g^3 - t_g)/12$$
$$= Cb(t – 1)t(t + 1)/12$$

in which

$$C = 1 – \sum_{g,j} (t_{gj}^3 - t_{gj}) / \{b(t – 1)\}t(t + 1)\}.$$

Thus $S_A = S/C$.

For the Durbin test statistic suppose that on the jth block the gth group is of size $t_{g,j}$. The same approach finds that $D_A = D/C$ in which, as for the Friedman case,

$$C = 1 – \sum_{g,j} (t_{gj}^3 - t_{gj}) / \{b(t – 1)\}t(t + 1)\}.$$

Jams Example. Three plum jams, A, B and C were given JAR sweetness codes by eight judges. The codes ranged from 1 denotes not sweet enough to 5 too sweet. These data were analysed in Rayner and Best (2017, 2018). Table 1 gives the ranked data. Judges are blocks.

Using mid-ranks the rank sums are 12.5, 19.5 and 16, leading to Friedman test statistic of F = 49/16 = 3.0625. On six blocks there two ties, so that on these blocks $\sum_g (t_{gj}^3 - t_{gj})$ = 6, leading to an adjustment factor of D = 13/16 and an adjusted Friedman statistic $F_c = F/D$ = 49/13 = 3.7692. Alternatively, the sum of the squares of the mid-ranks is 109 and substituting in the formula in section 3 again leads to F_c = 3.7692.

The adjusted Friedman test has a χ_2^2 p-values 0.152. This compares with p-value 0.216 for the unadjusted Friedman test using the χ_2^2 distribution.

Table 1. Three jams ranked for sweetness by eight judges. Low JAR scores were ranked 1.

	Blocks (Judges)							
Jam	1	2	3	4	5	6	7	8
1	2.5	1.5	2.5	1	1.5	1	1	1.5
2	1	3	1	3	3	2.5	3	3
3	2.5	1.5	2.5	2	1.5	2.5	2	1.5

5. Discussion

Rayner (2020a) gives the CMH MS test for the CRD and shows the tests based on both *KW* and KW_A are in this class of tests.

Rayner (2020b) gives the CMH MS test for the RBD and shows the tests based on both *S* and S_A are in this class of tests.

For the BIBD a closed form for the CMH MS statistic isn't accessible, so no adjusted Durbin test can be found this way.

References

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RAYNER, J.C.W. (2020b). A Friedman Test Adjusted for Ties Using the Cochran–Mantel–Haenszel Mean Score Test. (Gives the CMH MS test for the RBD and the adjusted Friedman test and shows these are the same. Also gives the CMH MS test statistic in terms of the ANOVA F statistic.)

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