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On Bias-Robust Mean Squared Error Estimation for Pseudo-Linear Small Area Estimators

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Abstract

We discuss bias-robust mean squared error estimation for estimators of finite population domain means that can be expressed in pseudo-linear form, i.e. as weighted sums of sample values. Our approach represents an extension of the ideas in Royall and Cumberland (1978) and appears to lead to estimators that are simpler to implement, and potentially more robust, than those suggested in the small area literature. We illustrate the usefulness of our approach through extensive model-based and design-based simulation, with the latter based on two realistic survey data sets containing small area information.

Keywords Best linear unbiased prediction; M-quantile model; Random effects model; Small area estimation.

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1. INTRODUCTION

Linear models, and linear predictors based on these models, are widely used in survey-based inference. However, such models run the risk of misspecification, particularly with regard to second order and higher moments. Bias-robust methods for estimating the mean squared error (MSE) of linear predictors of finite population quantities, i.e. methods that remain approximately unbiased under failure of assumptions about second order and higher moments, have been developed. Valliant, Dorfman and Royall (2000, chapter 5) discuss bias-robust MSE estimation for such predictors when a population is assumed to follow a linear model.

In this paper we address a subsidiary problem, which is that of bias-robust MSE estimation for estimators of finite population domain means that can be expressed in pseudo-linear form, i.e. as weighted sums, but where the weights can depend on the sample values of the variable of interest. An important application, and one that motivates our approach, is small area inference. Consequently from now on we use 'small area' (or just 'area') to refer to a domain of interest. Our approach represents an extension of the ideas in Royall and Cumberland (1978) and appears to lead to simpler to implement MSE estimators than those that have been suggested in the small area literature, see Prasad and Rao (1990) and Rao (2003, section 6.2.6).

The structure of the paper is as follows. In section 2 we discuss area-specific MSE estimation under an area-specific linear model. We then show how our approach can be used for estimating the MSE of three different small area linear predictors when they are expressed in pseudo-linear form, (a) the empirical best linear unbiased predictor or EBLUP (Henderson, 1953); (b) the model-based direct estimator (MBDE) of Chandra and Chambers (2009); and (c) the M-quantile predictor (Chambers and Tzavidis, 2006). In section 3 we present results from a series of simulation studies that illustrate the model-based and the design-based properties of our approach to MSE estimation. Finally, in section 4 we summarize our main findings. Throughout, we use either i or h to index the D small areas of interest, and either j or k to index the distinct population units in these areas.

2. BIAS-ROBUST MSE ESTIMATION FOR PSEUDO-LINEAR ESTIMATORS

2.1 MSE Estimation under an Area-Specific Linear Model

When survey-based inference relates to the characteristics of a group of *D* areas that partition the surveyed population, it is usually not realistic to assume that a linear model that applies to the population as a whole also applies within each area. We therefore consider MSE estimation for estimators of area means when different linear models apply within different areas. In particular, we focus on estimators that can be expressed as weighted sums of the sample values, referring to them as 'linear' in what follows to indicate that they have a linear structure.

To start, let y_j denote the value of *Y* for unit *j* of the population and suppose that this unit is in area *i*. We assume an area-specific linear model for y_j of the form

$$y_j = x_j^T \beta_i + e_j \tag{1}$$

where x_j is a $p \times 1$ vector of unit level auxiliary variables for unit *j*, β_i is a $p \times 1$ vector of area-specific regression coefficients and e_j is a unit level random effect with mean zero and variance σ_j^2 that is uncorrelated between different population units. We do not assume anything about σ_j^2 at this point. Suppose also that there is a known number N_i of population units in area *i*, with n_i of these sampled. The total number of units in the population is $N = \sum_{i=1}^{D} N_i$, with corresponding total sample size $n = \sum_{i=1}^{D} n_i$. In what follows, we use *s* to denote the collection of units in sample, with s_i the subset drawn from area *i*, and use expressions like $j \in i$ and $j \in s$ to refer to the units making up area *i* and sample *s* respectively. Note that throughout this paper we assume that the sampling method used is uninformative for the population values of *Y* given the corresponding values of the auxiliary variables and knowledge of the area affiliations of the population units. As a consequence, (1) applies at both sample and population level.

Let y_s denote the vector of sample values of the y_j and let $w_{is} = \left\{ w_{ij}; j \in s \right\}$ denote a set of 'fixed' weights such that $\hat{m}_i = w_{is}^T y_s = \sum_{j \in s} w_{ij} y_j$ is a consistent estimator of $m_i = N_i^{-1} \sum_{j \in i} y_j$ under simple random sampling within area *i*. By 'fixed' here we mean that these weights do not depend on the sample values of *Y*. Without loss of generality, we set $\sum_{j \in s} w_{ij} = 1$, so that $w_{ij} = O(n_i^{-1})$ for $j \in s_i$ and $w_{ij} = o(n_i^{-1})$ for $j \notin s_i$. Here s_i denotes the n_i sample units from area *i*. The bias of $\hat{m}_i$ under (1) is then

$$E(\hat{m}_i - m_i) = \left(\sum_{h=1}^{D} \sum_{j \in s_h} w_{ij} x_j^T \beta_h \right) - \bar{x}_i^T \beta_i . \tag{2}$$

Here $\bar{x}_i$ denotes the vector of average values of the auxiliary variables in area *i*. Similarly, the prediction variance of $\hat{m}_i$ under (1) is

$$Var(\hat{m}_i - m_i) = N_i^{-2} \left\{ \sum_{h=1}^{D} \sum_{j \in s_h} a_{ij}^2 \sigma_j^2 + \sum_{j \in r_i} \sigma_j^2 \right\} \tag{3}$$

where r_i denotes the non-sampled units in area *i* and $a_{ij} = N_i w_{ij} - I(j \in i)$. We use $I(A)$ to denote the indicator function for event *A*, so $I(j \in i)$ takes the value 1 if population unit *j* is from area *i* and is zero otherwise. Note that since a_{ij} is $O(N_i n_i^{-1})$ for $j \in s_i$, the first term within the braces in (3) is the leading term of this prediction variance if N_i is large compared to n_i.

Let sample unit j be from area *h*. We consider the important special case where $\mu_j = E(y_j \mid x_j) = x_j^T \beta_h$ is estimated by $\hat{\mu}_j = \sum_{k \in s} \phi_{kj} y_k$, where the ϕ_{kj} are suitable weights. Then

$$y_j - \hat{\mu}_j = (1 - \phi_{jj}) y_j - \sum_{k \in s(-j)} \phi_{kj} y_k$$

and so

$$Var(y_j - \hat{\mu}_j) = \sigma_j^2 \left\{ (1 - \phi_{jj})^2 + \sum_{k \in s(-j)} \phi_{kj}^2 (\sigma_k^2 / \sigma_j^2) \right\} \tag{4}$$

under (1). Here $s(-j)$ denotes the sample s with unit j excluded. If in addition $\hat{\mu}_j$ is unbiased for μ_j under (1), i.e.

$$E(y_j - \hat{\mu}_j) = 0 \quad (5)$$

we can adopt the approach of Royall and Cumberland (1978) and estimate (3) by

$$\hat{V}(\hat{m}_i) = N_i^{-2}\left\{\sum_{h=1}^{D}\sum\nolimits_{j\in s_h} a_{ij}^2 \hat{\lambda}_j^{-1}(y_j - \hat{\mu}_j)^2 + \sum\nolimits_{j\in r_i}\hat{\sigma}_j^2\right\} \quad (6)$$

where

$$\hat{\lambda}_j = (1-\phi_{jj})^2 + \sum\nolimits_{k\in s(-j)}\hat{\gamma}_{kj}\phi_{kj}^2$$

and $\hat{\gamma}_{kj} = \hat{\sigma}_k^2 / \hat{\sigma}_j^2$. Usually, the estimates $\hat{\sigma}_j^2$ of the residual variances in (6) are derived under a 'working model' refinement to (1). In the situation of most concern to us, where the sample sizes within the different areas are too small to reliably estimate area-specific variability, a pooling assumption can be made, i.e. $\sigma_j^2 = \sigma^2$, in which case we put

$$\hat{\sigma}_j^2 = \hat{\sigma}^2 = n^{-1}\sum\nolimits_{j\in s}\left\{(1-\phi_{jj})^2 + \sum\nolimits_{k\in s(-j)}\phi_{kj}^2\right\}^{-1}(y_j - \hat{\mu}_j)^2 \quad (7)$$

and so (6) becomes

$$\hat{V}(\hat{m}_i) = N_i^{-2}\sum\nolimits_{j\in s}\left\{a_{ij}^2 + (N_i - n_i)n^{-1}\right\}\hat{\lambda}_j^{-1}(y_j - \hat{\mu}_j)^2 \quad (8)$$

where now $\hat{\lambda}_j = (1-\phi_{jj})^2 + \sum\nolimits_{k\in s(-j)}\phi_{kj}^2$. Since any assumptions regarding σ_j^2 in the working model extension of (1) only affect second order terms in (3), the estimator (8) is bias-robust, i.e. it remains approximately unbiased under misspecification of the second order moments of this working model.

A corresponding estimator of the MSE of $\hat{m}_i$ under (1) follows directly. This is

$$\hat{M}(\hat{m}_i) = \hat{V}(\hat{m}_i) + \hat{B}^2(\hat{m}_i). \quad (9)$$

where

$$\hat{B}(\hat{m}_i) = \sum_{h=1}^{D}\sum\nolimits_{j\in s_h} w_{ij}\hat{\mu}_j - N_i^{-1}\sum\nolimits_{j\in i}\hat{\mu}_j \quad (10)$$

is the obvious unbiased estimator of (2).

Use of the square of the unbiased estimator (10) of the bias of $\hat{m}_i$ in the mean squared error estimator (9) can be criticised because this term is not itself unbiased for the squared bias term in the mean squared error. This can be corrected by replacing (9) by

$$\hat{M}(\hat{m}_i) = \hat{V}(\hat{m}_i) + \hat{B}^2(\hat{m}_i) - \hat{V}\left\{\hat{B}(\hat{m}_i)\right\}. \quad (11)$$

where $\hat{V}\left\{\hat{B}(\hat{m}_i)\right\}$ is a consistent estimator of the variance of (10). However, small area sample sizes may lead to this estimate becoming quite unstable, and so users may still prefer (9) over (11). Obviously (9) is then a conservative estimator of the MSE of $\hat{m}_i$ under (1).

2.2 MSE Estimation for Pseudo-Linear Small Area Estimators

The approach to MSE estimation outlined in the previous sub-section assumed that the weights defining the linear estimator $\hat{m}_i$ do not depend on the sample values of *Y*. However, most small area estimators do not satisfy this condition, in the sense that they are pseudo-linear in structure, with weights that do depend on these sample values. For example, the Best Linear Unbiased Predictor (BLUP) of m_i under the linear mixed model variant of (1) where the area-specific regression parameters β_i are independent and identically distributed realisations of a random variable with expected value β and covariance matrix $\mathbf{\Gamma}$, can be written as a weighted sum of the sample values of *Y* where the weights depend on $\mathbf{\Gamma}$ (see Royall, 1976). Consequently, the empirical version of this predictor, the widely used EBLUP, is computed by substituting an efficient sample estimate of $\mathbf{\Gamma}$ (e.g. the REML estimate) into the BLUP weights. If the linear mixed model assumption is true, this sample estimator of $\mathbf{\Gamma}$ converges to the true value and consequently the EBLUP weights converge to the BLUP weights. That is, for large values of the overall sample size *n*, we can treat the EBLUP weights as fixed and use the MSE estimator (9) for the EBLUP. Of course, the EBLUP weights are <u>not</u> really fixed, and so (9) is therefore an approximation to the true MSE of the EBLUP that ignores the contribution to this MSE arising from the variability in estimation of $\mathbf{\Gamma}$. However, this potential underestimation needs to be balanced against the bias robustness of (9) under misspecification of the second order moments of *Y*, and may well lead to this MSE estimator being preferable to other MSE estimators for the EBLUP based on higher order approximations that depend on the linear mixed model being true, e.g. the estimator of Prasad and Rao (1990).

An important advantage of (9) is its wide applicability. Many small area estimators developed under models that are variants of (1) can be written in pseudo-linear form, i.e. as weighted sums of the sample values of *Y*. To illustrate, we now focus on three such estimators: the EBLUP (Rao, 2003, chapter 6), the Model-Based Direct Estimator (MBDE) of Chandra and Chambers (2009) and the M-quantile predictor of Chambers and Tzavidis (2006). Each of these estimators can be written in pseudo-linear form, with weights that satisfy $w_{ij} = O(n_i^{-1})$ for $j \in s_i$ and $w_{ij} = o(n_i^{-1})$ for $j \notin s_i$, and so (9) can be used.

2.2.1 MSE estimation for the EBLUP

We first consider the well-known EBLUP for m_i based on a unit level linear mixed model extension of (1) of the form

$$y_i = \mathbf{X}_i \beta + \mathbf{Z}_i u_i + e_i \qquad (12)$$

where y_i is the N_i-vector of population values of y_j in area *i*, $\mathbf{X}_i$ is the corresponding $N_i \times p$ matrix of auxiliary variable values x_j, $\mathbf{Z}_i$ is the $N_i \times q$ component of $\mathbf{X}_i$ corresponding to the *q* random components of β, u_i is the associated *q*-vector of area-specific random effects and e_i is the N_i-vector of individual random effects. It is typically assumed that the area and individual effects are mutually independent, with the area effects independently and identically distributed as $N(0,\Omega)$ and the individual effects independently and identically distributed as $N(0,\sigma^2)$. See Rao (2003, chapter 6) for development of the underlying theory of this predictor. We note that the EBLUP can be written in pseudo-linear form,

$$\hat{m}_i^{EBLUP} = \sum_{j \in s} w_{ij}^{EBLUP} y_j = (w_{is}^{EBLUP})^T y_s \qquad (13)$$

where

$$\mathbf{w}_{is}^{EBLUP} = (w_{ij}^{EBLUP}) = N_i^{-1}\left[\Delta_{is} + \left\{\hat{\mathbf{H}}_s^T\mathbf{X}_r^T + \left(\mathbf{I}_n - \hat{\mathbf{H}}_s^T\mathbf{X}_s^T\right)\hat{\Sigma}_{ss}^{-1}\hat{\Sigma}_{sr}\right\}\Delta_{ir}\right].$$

Here Δ_{ir} is the vector of size $N-n$ that 'picks out' the non-sampled units in area *i*, $\mathbf{X}_s$ and $\mathbf{X}_r$ are the matrices of order $n\times p$ and $(N-n)\times p$ respectively of the sample and non-sample values of the auxiliary variables, $\mathbf{I}_n$ is the identity matrix of order *n*, $\hat{\mathbf{H}}_s = \left(\mathbf{X}_s^T\hat{\boldsymbol{\Sigma}}_{ss}^{-1}\mathbf{X}_s\right)^{-1}\mathbf{X}_s^T\hat{\boldsymbol{\Sigma}}_{ss}^{-1}$, $\hat{\Sigma}_{ss} = \hat{\sigma}^2\mathbf{I}_n + diag\left\{\mathbf{Z}_{is}\hat{\Omega}\mathbf{Z}_{is}^T; i=1,\ldots,D\right\}$ and $\hat{\Sigma}_{sr} = diag\left\{\mathbf{Z}_{is}\hat{\Omega}\mathbf{Z}_{ir}^T; i=1,\ldots,D\right\}$. Here $\mathbf{Z}_{is}$ ($\mathbf{Z}_{ir}$) is the sample (non-sample) component of $\mathbf{Z}_i$ and $\hat{\sigma}^2$ and $\hat{\Omega}$ are suitable (e.g. ML or REML) estimates of the variance components of (12).

Given this setup, estimation of the area-specific MSE of the EBLUP can be carried out using (9) with weights defined following (13). In turn, this requires that we have access to unbiased estimators $\hat{\mu}_j$ of the area specific individual expected values μ_j. As we have already noted, such estimators may be unstable when area sample sizes are small. Consequently, it is tempting to replace $\hat{\mu}_j$ by the EBLUP for y_j, i.e. $\hat{y}_j^{EBLUP} = x_j^T\hat{\beta}^{EBLUE} + z_j^T\hat{u}_i^{EBLUP}$, where $\hat{\beta}^{EBLUE}$ denotes the Empirical Best Linear Unbiased Estimator of β in the linear mixed model (12) and $\hat{u}_i^{EBLUP}$ denotes the predicted area effect for the area *i* that contains observation *j*. Unfortunately, because of the well-known shrinkage effect associated with EBLUPs, this approach is not recommended. To illustrate this, we note that $\hat{V}(\hat{m}_i)$ in (9) uses $(y_j - \hat{\mu}_j)^2$ as an estimator of $E(y_j-\mu_j)^2$. The bias in this estimator is therefore

$$\begin{aligned}E(y_j-\hat{\mu}_j)^2 - E(y_j-\mu_j)^2 &= -2E(y_j-\mu_j)(\hat{\mu}_j-\mu_j) + E(\hat{\mu}_j-\mu_j)^2\\ &= -E\left\{(\hat{\mu}_j-\mu_j)(2y_j-\mu_j-\hat{\mu}_j)\right\}\end{aligned}$$

so we anticipate that $\hat{V}(\hat{m}_i)$ will be negatively biased if $E\left\{(\hat{\mu}_j-\mu_j)(2y_j-\mu_j-\hat{\mu}_j)\right\}$ is positive and vice versa. Now let sample unit *j* be from area *i* and consider the special case of a random intercept model for y_j, i.e. $y_j = x_j^T\beta + u_i + e_j$ where u_i is the random effect for area *i* and e_j is a random individual effect uncorrelated with u_i. Here $\mu_j = x_j^T\beta + u_i$. Suppose that we have a large overall sample size, allowing us to replace $\hat{\beta}^{EBLUE}$ by β. The EBLUP $\hat{\mu}_j = \hat{y}_j^{EBLUP}$ can then be approximated by $\tilde{\mu}_j = x_j^T \beta + \gamma_i u_i$, where γ_i is a 'shrinkage' factor. It follows that

$$(\tilde{\mu}_j - \mu_j)(2y_j - \mu_j - \tilde{\mu}_j) = 2u_i(\gamma_i - 1)e_i - u_i^2(\gamma_i - 1)^2$$

so $E(y_j - \hat{\mu}_j)^2 - E(y_j - \mu_j)^2 \approx (\gamma_i - 1)^2 \sigma_u^2$. That is, we expect $\hat{V}(\hat{m}_i)$ to be positively biased if we use the shrunken EBLUP $\hat{y}_j^{EBLUP}$ to define $\hat{\mu}_j$. We also note that this bias disappears (approximately) if we 'unshrink' the residual component of this EBLUP. For example, in the case of the popular random intercepts model, we use

$$\hat{\mu}_j = x_j^T \hat{\beta}^{EBLUE} + (\bar{y}_{is} - \bar{x}_{is}^T \hat{\beta}^{EBLUE}) = \bar{y}_{is} + (x_j - \bar{x}_{is})^T \hat{\beta}^{EBLUE}$$

where $\bar{y}_{is}$ and $\bar{x}_{is}$ denote the sample means of *Y* and *X* respectively in area *i*. Given (12) is the working model, a general expression for such an 'unshrunken' estimator is

$$\hat{\mu}_j = x_j^T \hat{\beta}^{EBLUE} + z_j^T \tilde{u}_i \tag{14}$$

where $\tilde{u}_i = \left(\mathbf{Z}_{is}^T \mathbf{Z}_{is}\right)^{-1} \mathbf{Z}_{is}^T \left(y_{is} - \mathbf{X}_{is} \hat{\beta}^{EBLUE}\right)$ is the unshrunken predictor of the random effect for area *i*. It is not difficult to see that then $\hat{\mu}_j = \sum_{k \in s} \phi_{kj} y_k$ where $\phi_{kj} = c_{ijsk} + d_{ijsk} I(k \in i)$, with

$$\mathbf{c}_{ijs} = \left(c_{ijsk}; k \in s\right) = \hat{\Sigma}_{ss}^{-1} \mathbf{X}_s \left(\mathbf{X}_s^T \hat{\Sigma}_{ss}^{-1} \mathbf{X}_s\right)^{-1} \left\{x_j - \mathbf{X}_{is}^T \mathbf{Z}_{is} \left(\mathbf{Z}_{is}^T \mathbf{Z}_{is}\right)^{-1} z_j\right\}$$

and

$$\mathbf{d}_{ijs} = \left(d_{ijsk}; k \in s_i\right) = \mathbf{Z}_{is} \left(\mathbf{Z}_{is}^T \mathbf{Z}_{is}\right)^{-1} z_j .$$

Finally, we observe that when (14) is used in (9), the estimated bias (10) becomes

$$\hat{B}(\hat{m}_i) = \sum_{h=1}^{D} \left(\sum_{j \in s_h} w_{ij}^{EBLUP} z_j\right)^T \tilde{u}_h - \bar{z}_i^T \tilde{u}_i \tag{15}$$

since the EBLUP weights (13) are 'locally calibrated' on *X*, i.e. $\sum_{j \in s} w_{ij}^{EBLUP} x_j = \bar{x}_i$. Typically, (15) is close to zero and so the estimated MSE of the EBLUP based on (9) is essentially the estimate (8) of its prediction variance.

2.2.2 MSE estimation for the MBDE

The second predictor of m_i that we consider is the Model-Based Direct Estimator (MBDE) described in Chandra and Chambers (2009). This is based on the same linear mixed model (12) as the EBLUP, with the MBDE predictor defined as

$$\hat{m}_i^{MBDE} = \sum_{j \in s} w_{ij}^{MBDE} y_j = (w_{is}^{MBDE})^T y_s \tag{16}$$

where

$$w_{ij}^{MBDE} = \frac{I(j \in s_i) w_j^{EBLUP}}{\sum_{k \in s} I(k \in s_i) w_k^{EBLUP}} .$$

Here $I(j \in s_i)$ is the indicator function for unit *j* to be in the area *i* sample, and $w_s^{EBLUP} = (w_j^{EBLUP})$ is the vector of weights that defines the EBLUP for the population total of the y_j under (12), i.e.

$$w_s^{EBLUP} = (w_j^{EBLUP}) = 1_n + \left\{ \hat{\mathbf{H}}_s^T \mathbf{X}_r^T + \left(\mathbf{I}_n - \hat{\mathbf{H}}_s^T \mathbf{X}_s^T \right) \hat{\boldsymbol{\Sigma}}_{ss}^{-1} \hat{\boldsymbol{\Sigma}}_{sr} \right\} 1_{N-n} \tag{17}$$

where 1_n (1_{N-n}) denotes the unit vector of size n $(N-n)$ and $\hat{\mathbf{H}}_s$ was defined in section 2.2.1. In this case pseudo-linearisation based estimation of the area-specific MSE of the MBDE is carried out using (9), with weights defined following (16). Note that the estimated expected values used in (9) when applied to the MBDE are the same as the unshrunken estimates (14) used with the EBLUP, reflecting the fact that both the MBDE and the EBLUP are based on the same linear mixed model (12). However, the MBDE weights used in (16) are not locally calibrated, and so the squared bias term in (9) cannot be ignored when estimating the MSE of this predictor.

2.2.3 MSE estimation for the M-quantile estimator

The third estimator that we consider is based on the M-quantile modelling approach described in Chambers and Tzavidis (2006). This approach does not assume an underlying linear mixed model, relying instead on characterising the relationship between y_j and x_j in area *i* in terms of the linear M-quantile model that best 'fits' the sample y_j values from this area. That is, this approach replaces (12) by a model of the form

$$y_i = \mathbf{X}_i \beta(q_i) + e_i \quad (18)$$

where $\beta(q)$ denotes the coefficient vector of a linear model for the regression M-quantile of order *q* for the population values of *Y* and **X**, and q_i denotes the M-quantile coefficient of area *i*. Given an estimate $\hat{q}_i$ of q_i, an iteratively reweighted least squares (IRLS) algorithm is used to calculate an estimate

$$\hat{\beta}(\hat{q}_i) = \left\{ \mathbf{X}_s' \mathbf{W}_s(\hat{q}_i) \mathbf{X}_s \right\}^{-1} \mathbf{X}_s' \mathbf{W}_s(\hat{q}_i) y_s \quad (19)$$

of $\beta(q_i)$, and a non-sample value of y_j in area *i* is then predicted by $\hat{y}_j = x_j^T \hat{\beta}(\hat{q}_i)$. Here $\mathbf{W}_s(\hat{q}_i)$ is the diagonal matrix of final weights used in the IRLS algorithm.

Tzavidis, Marchetti and Chambers (2009) note that value of the M-quantile estimator suggested in Chambers and Tzavidis (2006) can be interpreted as the expected value of *Y* in area *i* with respect to a biased estimator of the distribution function of this variable in the area. They therefore develop an improved M-quantile estimator, replacing this biased distribution function estimator by the Chambers and Dunstan (1986) distribution function estimator under the area-specific model (1). This corresponds to predicting m_i by

$$\hat{m}_i^{MQ} = \sum_{j \in s} w_{ij}^{MQ} y_j = (w_{is}^{MQ})^T y_s \quad (20)$$

where

$$w_{is}^{MQ} = n_i^{-1} \Delta_{is} + (1 - N_i^{-1} n_i) \mathbf{W}_s(\hat{q}_i) \mathbf{X}_s \left\{ \mathbf{X}_s^T \mathbf{W}_s(\hat{q}_i) \mathbf{X}_s \right\}^{-1} (\bar{x}_{ir} - \bar{x}_{is}) .$$

Here $\bar{x}_{is}$ and $\bar{x}_{ir}$ are the vectors of sample and non-sample means of the x_j in area *i*. It is not difficult to show that the weights following (20) are locally calibrated. Furthermore, if we then put $\hat{\mu}_j = x_j^T \hat{\beta}(\hat{q}_i)$ it is easy to see that (10) is zero and so the area-specific MSE of the bias-corrected M-quantile estimator (20) can be estimated using just the estimated prediction variance component (8). Since the constant $\hat{\lambda}_j$ in (8) is typically very close to one under M-quantile estimation, we set it equal to this value whenever we compute values of (8) that relate to small area estimation under the M-quantile modelling approach.

As we have already done with the EBLUP, we note that use of (8) implicitly treats the weights defining (20) as fixed, which is actually not the case since the matrix $\mathbf{W}_s(\hat{q}_i)$ is a function of the sample values of *Y*. An immediate consequence is that pseudo-linearisation based estimation of the MSE of the M-quantile predictor via (8) is a first order approximation to the true MSE of this estimator. Nevertheless, since accounting for weight variability in the definition of the M-quantile estimator considerably complicates estimation of its MSE - see Street, Carroll and Ruppert (1988) for an examination of this issue in the context of 'standard' M-estimation of regression coefficients - it is of interest to see how the relatively simple estimator (8) performs when used to estimate this MSE.

2.3 MSE Estimation for the Pseudo-Linear Synthetic EBLUP

In many small area applications there are areas that contain no sample, and hence synthetic estimation is used. Without loss of generality, we assume that these areas are numbered last, i.e. if D^+ areas have non-zero sample then $n_h > 0$ for $h \le D^+$ and $n_h = 0$ for $h > D^+$. For $i > D^+$ the 'synthetic EBLUP' for m_i is

$$\hat{m}_i^{SYN-EBLUP} = \bar{x}_i^T \hat{\beta}^{EBLUP} = \left(w_s^{EBLUP-i}\right)^T y_s = \sum_{h=1}^{D^+} \sum\nolimits_{j \in s_h} w_{ij}^{EBLUP} y_j \tag{21}$$

where

$$w_{is}^{EBLUP} = \left(w_{ij}^{EBLUP}\right) = \hat{\mathbf{H}}_s^T \bar{x}_i .$$

Clearly (21) is a pseudo-linear estimator, and so we can use (8) to estimate its prediction variance, observing that since $n_i = 0$, $a_{ij} = N_i w_{ij}^{EBLUP}$ and so (8) becomes

$$\hat{V}(\hat{m}_i^{SYN-EBLUP}) = \sum\nolimits_{j \in s} \left\{ \left(w_{ij}^{EBLUP}\right)^2 + N^{-1} n^{-1} \right\} \hat{\lambda}_j^{-1} (y_j - \hat{\mu}_j)^2 . \tag{22}$$

Unfortunately, since there is no sample in area *i*, we cannot use (10) to estimate the area-specific bias (2) of $\hat{m}_i^{SYN-EBLUP}$. However, under the linear mixed model (12), this bias has expected value

$$E(\hat{m}_i^{SYN-EBLUP} - m_i) = \sum_{h=1}^{D^+} \sum_{j \in s_h} w_{ij}^{EBLUP} \left(x_j^T \beta + z_j^T u_h \right) - \overline{x}_i^T \beta - \overline{z}_i^T u_i .$$

The conditional expectation of the square of this expected bias, given the area effects $u_s = (u_h; h = 1, \ldots, D^+)$ for the sampled areas, is

$$E\left\{ E^2(\hat{m}_i^{SYN-EBLUP} - m_i) \middle| \mathbf{X}, u_s \right\} = \left\{ \sum_{h=1}^{D^+} \sum_{j \in s_h} w_{ij}^{EBLUP} \left(x_j^T \beta + z_j^T u_h \right) - \overline{x}_i^T \beta \right\}^2 + \overline{z}_i^T \Omega \overline{z}_i$$

which immediately suggests that for a non-sampled area *i* we estimate the squared bias of the synthetic estimator $\hat{m}_i^{SYN-EBLUP}$ by

$$\hat{B}^2(\hat{m}_i^{SYN-EBLUP}) = \left\{ \sum_{h=1}^{D^+} \sum_{j \in s_h} w_{ij}^{EBLUP} \left(x_j^T \hat{\beta}^{EBLUE} + z_j^T \tilde{u}_h \right) - \overline{x}_i^T \hat{\beta}^{EBLUE} \right\}^2 + \overline{z}_i^T \hat{\Omega} \overline{z}_i . \quad (23)$$

Here $\tilde{u}_h$ is the 'unshrunken' estimated effect for sampled area *h* – see (14). Our proposed MSE estimator for $\hat{m}_i^{SYN-EBLUP}$ is then the sum of (22) and (23). Note that, unlike (9), this MSE estimator includes no information from area *i*, and so is not an estimator of the area-specific MSE of (21). In particular, its validity depends completely on the mixed model (12) holding, and so it is not robust to misspecification of this model.

3. SIMULATION STUDIES OF THE PROPOSED MSE ESTIMATOR

In this section we describe results from five simulation studies that aim at assessing the performance of the approach to robust MSE estimation described in the previous section. Three of these studies are model-based simulations, with population data generated from the linear mixed model (12). The remaining two are design-based simulations, with population data derived from two real survey datasets where linear small area estimation is of interest.

Given our focus on bias-robustness, the main performance indicator for an MSE estimator in all five studies is its median relative bias, defined by

$$RB(M) = \underset{i}{median}\left\{ M_i^{-1}K^{-1}\sum_{k=1}^{K}\left(\hat{M}_{ik} - M_i\right)\right\} \times 100 .$$

Here the subscript i indexes the small areas and the subscript k indexes the K Monte Carlo simulations, with $\hat{M}_{ik}$ denoting the simulation k value of the MSE estimator in area i, and M_i denotes the actual (i.e. Monte Carlo) MSE in area i. Since we would naturally prefer to use the more stable of two unbiased MSE estimators, we also measured the stability of an MSE estimator by its median percent relative root mean squared error,

$$RRMSE(M) = \underset{i}{median}\left\{ \sqrt{K^{-1}\sum_{k=1}^{K}\left(\frac{\hat{M}_{ik} - M_i}{M_i}\right)^2} \right\}.$$

Although the purpose of this paper is not to compare different methods of small area estimation, it is useful to relate MSE estimation performance for a particular method of small area estimation to the actual estimation performance of this method. We therefore provide two measures of the relative performance of the three small area estimation methods (EBLUP, MBDE and M-quantile) that were used in our simulations. These are the median percent relative bias

$$RB(m) = \underset{i}{median}\left\{ \bar{m}_i^{-1}K^{-1}\sum_{k=1}^{K}\left(\hat{m}_{ik} - m_{ik}\right)\right\} \times 100$$

and the median percent relative root mean squared error

$$RRMSE(m) = \underset{i}{median}\left\{ \sqrt{K^{-1}\sum_{k=1}^{K}\left(\frac{\hat{m}_{ik} - m_{ik}}{m_{ik}}\right)^2} \right\} \times 100$$

of the estimates $\hat{m}_{ik}$ generated by an estimation method. Note that $\bar{m}_i = K^{-1}\sum_{k=1}^{K} m_{ik}$ here.

3.1 Model-Based Simulations

The first model-based simulation study was based on population data generated under the mixed model (12) with Gaussian random effects. It used a population size of N = 15,000, with D = 30 small areas. Population sizes in the small areas were uniformly distributed over the interval [443, 542] and kept fixed over simulations. At each simulation, population values for Y were generated under the random intercepts model $y_j = 500 + 1.5x_j + u_i + e_j$, with x_j drawn from a chi squared distribution with 20 degrees of freedom. The area effects u_i and individual effects e_j were independently drawn from $N(0,\sigma_u^2)$ and $N(0,\sigma_e^2)$ distributions respectively, with the values of σ_u and σ_e shown in rows SIM1-A and SIM1-B of Table 2. A sample of size $n = 600$ was selected from each simulated population, with area sample sizes proportional to the fixed area populations, resulting in an average area sample size of $n_i = 20$. Sampling was via stratified random sampling, with the strata defined by the small areas. A total of K = 1000 simulations were carried out.

Conditions for the second model-based simulation study were the same as in the first, with the exception that the area level random effects and the individual level random effects were independently drawn from mean corrected chi-square distributions respectively. The corresponding values of the area level and individual level variances are shown in rows SIM2-A and SIM2-B in Table 2. Finally, in the third model-based simulation study conditions were kept the same as in the first for areas 1 – 25, but in areas 26 – 30 area effects were independently drawn from a normal distribution with a larger variance. We refer to this as a Mixture in Table 2, with variances for the two sets of areas shown in rows SIM3-A and SIM3-B. None of the three small area estimation methods that we consider here claim to be robust to the area-wide outlier behaviour simulated in our third study, and so it is important that their corresponding MSE estimators react to it by tracking the resulting increase in variability in the 'outlier areas'. We therefore only show values relating to areas 26 – 30 in the results for this simulation reported in Tables 3 and 4. We also replicated all three scenarios above using reduced overall sample sizes of $n = 300$ (with average area sample size $n_i = 10$) and $n = 150$ (with average area sample size $n_i = 5$). These additional simulations allowed us to investigate the effect of reduced sample sizes on the performance of the MSE estimators.

Table 3 shows the median bias *RB(m)* and median relative root mean squared error *RRMSE(m)* of the small area estimation methods investigated in our simulations for the three sample sizes (n =600, 300 and 150). These are the direct estimator (i.e. the small area mean), the EBLUP with weights defined by (13), the MBDE with weights defined by (16) and the M-quantile estimator defined by the weights (20). In Table 4 we show the corresponding performances of MSE estimators for these small area estimators. Note that we provide results for four MSE estimators for the EBLUP, with PR0 denoting the estimator suggested by Prasad and Rao (1990), see Rao (2003, section 6.2.6). It is noteworthy that PR0 is not an estimator of the area-specific MSE of the EBLUP, but of its MSE under the mixed linear model (12), i.e. averaged over possible realisations of the area effect. In contrast, the MSE estimators PR1 and PR2 in Table 4 are the area specific versions of PR0 suggested in Rao (2003, section 6.3.2 expressions 6.3.15 and 6.3.16 respectively), while the Robust estimator is the estimator of the area-specific MSE of the EBLUP defined in Table 1. Similarly, the Robust estimators of the MSE of the MBDE and the M-quantile estimator are defined by their corresponding area-specific entries in Table 1. The MSE estimator of the Direct estimator is its usual variance estimator under simple random sampling without replacement, which we denote by SRS.

The differences between the various estimators in Table 3 are essentially as one would expect. Bias is not really an issue (to be expected given the population data follow a linear model in all cases), while the indirect estimators (EBLUP and M-quantile) are more efficient than the direct estimators (Direct and MBDE), with the M-quantile estimator the best performer in the two mixture-based simulations (SIM3-A and SIM3-B). Note that in this case the M-quantile weights (20) are based on an outlier-robust estimate of the M-quantile coefficient $\hat{q}_i$ for area *i*, defined by the median (rather than the mean) of the M-quantile coefficients of sampled units in this area. Further, as the sample sizes decrease, the RRMSEs of both the direct and the indirect estimators increase, but their relative performances remain the same. Under normality the EBLUP is better than the M-quantile estimator but the differences between these two estimators become smaller as we move away from normality, with the M-quantile estimator more efficient in the mixture model scenarios.

The results set out in Table 4 focus on the biases of the various MSE estimators that we considered. To start, we note that the relative performances of these MSE estimators with respect to bias do not change with decreasing sample size. Not surprisingly, given that all its underlying assumptions are met, the PR0 estimator and its area-specific alternatives, PR1 and PR2, perform very well in both normal scenarios (SIM1-A and SIM1-B) and both chi-squared scenarios (SIM2-A and SIM2-B), with virtually no bias except when within area sample sizes are very small. The same applies to the SRS estimator. When applied to the EBLUP, the Robust MSE estimator on the other hand shows positive bias under both the normal and chi-squared scenarios, particularly for moderate intra-cluster correlation (SIM1A and SIM2A), which increases with decreasing sample size. However, things change when we examine the results for the mixture model scenarios (SIM3-A and SIM3B). Here we see a substantial negative bias for all three versions of PR. In comparison, the Robust MSE estimator for the EBLUP now shows a much smaller negative bias while the same MSE estimator applied to the M-quantile estimator shows an upward bias. The Robust MSE estimators for the direct estimators (MBDE and Direct) are essentially unbiased. Given that as far as MSE estimation is concerned, positive bias is preferable to negative bias, it seems clear that the Robust MSE estimator is better able to handle this outlier situation. Figure 1 graphically illustrates this point. Here we show the area-specific RMSEs and the average (over the simulations) of the estimated RMSEs in each of the 30 areas for the mixture simulation SIM3-A, with the vertical line delineating the last five 'outlier' areas. In the top panel of this plot we can see that the PR0 estimator is unable to detect the step increase in the MSE of the EBLUP for these 'outlier' areas, being biased slightly high in the 'well-behaved' areas and then biased rather low in the 'outlier' areas. In contrast, the Robust MSE estimator for the EBLUP and the MBDE tracks the area specific RMSEs rather well, while the same MSE estimator based on M-quantile weights tends to be biased low in the 'well-behaved' areas, and biased high in the 'outlier' areas, which can be argued as being perhaps a rather better outcome than that recorded by the PR0 estimator in this simulation.

Finally, Table 5 shows the relative RMSEs of the different MSE estimators across the three types of model-based simulation, allowing one to compare these MSE estimators on the basis of their relative stability. Here we see that bias-robust MSE estimation comes at a price. In particular, all three versions of the PR estimator of the MSE of the EBLUP are more stable than the Robust MSE estimator of the EBLUP. The same is true of the Robust MSE estimators for the MBDE and the M-quantile estimator. Essentially, given that the population data follow a mixed linear model, the PR estimator of MSE is very stable, while the Robust MSE estimator is more variable.

Although all methods of MSE estimation that we evaluated (with the exception of SRS) exhibited some bias for very small area sample sizes, our model-based simulation results provide evidence that the robust MSE estimation method (9) is bias robust when applied to the three pseudo-linear small area estimators EBLUP, MBDE and M-quantile. In contrast, and as one might expect, the model dependent 'area-averaged' MSE estimator PR0 for the EBLUP exhibits bias under model failure. The fact that we observed rather similar behaviour for the area-specific versions PR1 and PR2 of this MSE estimator indicates that 'area specific' does not necessarily mean 'bias robust'. Our results also show that the bias robust MSE estimator (9) can be much more variable than the model dependent PR estimators that we investigated, so there is a clear efficiency price for this robustness.

3.2 Design-Based Simulations

What happens when, as in real life, we cannot be confident that our data follow a linear mixed model? In order to investigate this situation, we report results from two design-based simulation studies, both based on realistic populations, where a linear model assumption is essentially an approximation. The first involved a sample of 3591 households spread across D = 36 districts of Albania that participated in the 2002 Albanian Living Standards Measurement Study. This sample was bootstrapped to create a realistic population of $N = 724,782$ households by re-sampling with replacement with probability proportional to a household's sample weight. A total of $K = 1000$ independent stratified random samples were then drawn from this bootstrap population, with total sample size equal to that of the original sample and with districts defining the strata. Sample sizes within districts were the same as in the original sample, and varied between 8 and 688 (with median district sample size equal to 56). The Y variable of interest was household per capita consumption expenditure (HCE) and X was defined by three zero-one variables (ownership of television, parabolic antenna and land). The aim was to estimate the average value of HCE for each district. In the original 2002 survey, the linear relationship between HCE and the three variables making up X was rather weak, with very low predictive power. In particular, only ownership of land was significantly related to HCE at the five percent level. This fit was considerably improved by extending the linear model to include random intercepts, defined by independent district effects. These explained approximately 10 per cent of the residual variation in this model.

The second design-based simulation study was based on the same population of Australian broadacre farms as that used in the simulation studies reported in Chambers and Tzavidis (2006) and Chandra and Chambers (2009). This population was defined by bootstrapping a sample of 1652 farms that participated in the Australian Agricultural and Grazing Industries Survey (AAGIS) to create a population of $N = 81,982$ farms by re-sampling from the original AAGIS sample with probability proportional to a farm's sample weight. The small areas of interest in this case were the D = 29 broadacre farming regions represented in this sample. The design-based simulation was carried out by selecting $K = 1000$ independent stratified random samples from this bootstrap population, with strata defined by the regions and with stratum sample sizes defined by those in the original AAGIS sample. These sample sizes vary from 6 to 117, with a median region sample size of 55. Here Y is Total Cash Costs (TCC) associated with operation of the farm, and X is a vector that includes farm area (Area), effects for six post-strata defined by three climatic zones and two farm size bands as well as the interactions of these variables. In the original AAGIS sample the relationship between TCC and Area varies significantly between the six post-strata, with an overall Rsquared value of approximately 0.48 after the deletion of two outliers. The fixed effects in the prediction model were therefore specified as corresponding to a separate linear fit of TCC in terms of Area in each post-stratum. Random effects (necessary for computation of the EBLUP and the

MBDE, but not the M-quantile predictor) were defined as independent regional effects (i.e. a random intercepts specification) on the basis that in the original AAGIS sample the between region variance component is highly significant, explaining just over 10 per cent of the total residual variability with the two outliers removed. A slightly more efficient random effects specification, involving a random slope on the Area term in the model, can be used when modelling TCC in terms of Area in the AAGIS data, but was felt to be too sensitive in terms of inducing instability in the EBLUP. The aim was to estimate the regional averages of TCC.

Tables 6 and 8 show the median relative biases and the median relative RMSEs of different estimators based on the $K = 1000$ independent stratified samples taken from the Albanian and AAGIS populations respectively. Similarly, Tables 7 and 9 show the median relative biases and median relative RMSEs of corresponding estimators of the MSEs of these estimators calculated from the same samples. It is noteworthy that in spite of the fact that the mixed linear models fitted to both the Albanian and AAGIS data appear reasonable, the gains from adoption of small area estimation methods based on them do not lead to substantial improvements in efficiency given the original small area sample sizes for these surveys. In particular, the M-quantile estimator, which is not based on a random effects specification, works best overall in terms of both bias and MSE, while the EBLUP, although the best performer in terms of MSE for the Albanian population, is also the worst for the AAGIS population and records the highest biases in both.

Design-based simulations based on the Albanian and AAGIS populations were also carried out using smaller regional sample sizes than in the original surveys. In particular, the overall sample size was reduced for the Albanian population to $n = 436$ (with a median district sample size of 11) and then to $n = 291$ (with a median district sample size of 9). Similarly, the overall sample size was reduced for the AAGIS population to $n = 327$ (with a median regional sample size of 12) and then to $n = 243$ (with a median regional sample size of 8). As expected the RMSE of the point estimators increases as the area sample sizes decrease. Overall, the EBLUP improves its RMSE performance relative to all other estimators for the Albanian population, and performs similarly to the M-quantile estimator for the AAGIS population, with smaller sample sizes. However, since the realism of these reduced sample size designs is somewhat questionable, we do not place too much emphasis on results derived from them, noting only that they are useful for assessing the performance of MSE estimators with realistic data and with very small sample sizes.

Reflecting their model-dependent basis, all three PR-based MSE estimators for the EBLUP display a substantial upward bias in both sets of design-based simulations as well as the largest instability under the original sample design. It is noteworthy that for the Albanian population at least the relative performances of these MSE estimators improves with smaller samples, but only because the Robust MSE estimators then become more unstable. For the AAGIS population the PR-based MSE estimators perform badly at all sample sizes. This corroborates comments by other authors (e.g. Longford, 2007) about the poor design-based properties of this estimator. In contrast, for the Albanian population all three versions of the Robust MSE estimator are essentially unbiased, while for the AAGIS population the Robust MSE estimator is unbiased for the MBDE and the M-quantile estimator and biased upwards for the EBLUP, though not to the same extent as the PR estimator of the MSE of the EBLUP.

An insight into the reasons for this difference in behaviour can be obtained by examining the area specific RMSE values displayed in Figure 2 for the Albanian population and in Figure 3 for the AAGIS population. Note that in both cases the sample sizes are those from the original surveys. Thus, in Figure 2 we see that all three Robust RMSE estimators track the district-specific design-based RMSEs of their respective estimators exceptionally well while the PR0 estimator does not seem to be able to capture between district differences in the design-based RMSE of the EBLUP. In contrast, in Figure 3 we see that the Robust estimator of the RMSE of the M-quantile predictor performs extremely well in all regions, with the corresponding estimator of the RMSE of the MBDE also performing well in all regions except one (region 6) where it substantially overestimates the design-based RMSE of this predictor. This region is noteworthy because samples that are unbalanced with respect to Area within the region lead to negative weights under the assumed linear mixed model. The picture becomes more complex when one considers the region-specific RMSE estimation performance of the EBLUP in Figure 3. Here we see that the Robust estimator of the RMSE of the EBLUP clearly tracks the region-specific design-based RMSE of this predictor better than the PR0 estimator, with the noteworthy exception of region 21, where it shows significant overestimation. This region contains a number of massive outliers (all replicated from a single outlier in the original AAGIS sample) and these lead to a 'blow out' in the value of Robust when they appear in sample (this can also be seen in the results for the MBDE and the M-quantile predictors for this region). In contrast, with the exception of region 6 (where sample balance is a problem), there seems to be little regional variation in the value of the PR0 estimator of the RMSE of the EBLUP, indicating a serious bias problem.

As noted earlier, it is not uncommon to want to produce an estimate for a small area where there is no sample. In such cases, one has to rely completely on the correctness of the model specification. In Tables 10 and 11 we illustrate the importance of this assumption by contrasting the estimation and MSE estimation performances of the EBLUP for sampled areas with that of the Synthetic EBLUP for areas where no sample data are available. Two situations are shown. The first is a modification of the model-based SIM1-A simulation with a small average sample size and with five zero-sample areas. The second is a similar small sample modification of the design-based simulation based on the AAGIS population, with four zero-sample areas. It is clear that when the model underpinning the EBLUP actually holds (i.e. SIM1-A), estimation and MSE estimation (either based on PR0, or on the Robust alternative) works well. The problem is that when there is some doubt about how well this model holds (as in the AAGIS population), then the EBLUP can fail, and our estimator of its MSE can also fail to identify this problem. This is nicely illustrated by the results for the AAGIS population in Tables 10 and 11 where we see that both the PR0 and Robust MSE estimators for the Synthetic EBLUP fail to identify the large positive bias of the Synthetic EBLUP and so end up with a large downward bias.

Finally, we provide information that allows one to assess the usefulness of the simulation-based comparisons shown in this paper, noting that the aim of these simulations is not to provide precise estimates of the properties of different estimators but to distinguish their relative performance. In particular, we investigated the stability of these comparisons by checking to see how much they changed between three different stages of the simulations i.e. after 250, 500 and 1000 simulations. Tables 12 and 13 display these results for the design-based simulations using the AAGIS population with a reduced sample size (median regional sample size of 12). Similar results, not provided here but available from the authors on request, were observed for the model-based simulations SIM1-A and SIM1-B with an average sample size of 5. Overall, we conclude that the number of simulations that we carried out is sufficient to distinguish the relative performances of the different small area estimators and MSE estimators that we focus on in this paper.

4. CONCLUSIONS AND DISCUSSION

In this paper we propose a bias-robust and easily implemented method of estimating the mean squared error of pseudo-linear estimators of small area means (and totals). The empirical results described in section 3 are evidence that this method has promise as a general-purpose approach. In particular, it performed reasonably well overall in terms of estimating both model-based and design-based MSE for the three rather different pseudo-linear estimators that we investigated in our simulations. This was in contrast to the more complex model-dependent approach underpinning the estimator of the MSE of the EBLUP suggested by Prasad and Rao (1990), which worked very well in terms of bias and overall stability when its model assumptions were valid (SIM1-A to SIM2-B in our model-based simulations) but then ran into bias problems in the presence of outlier area effects (SIM3-A and SIM3-B) and for both our fairly realistic design-based simulations where model fit could only be considered as approximately valid.

The Robust MSE estimator proposed in this paper can be easily applied to other pseudo-linear small area estimators where current approaches to MSE estimation are not straightforward. Two prominent examples are MSE estimation for the Pseudo-EBLUP (Prasad and Rao, 1999; You and

Rao, 2002) and for the model-assisted empirical best predictor described by Jiang and Lahiri (2006). Similarly, since the nonparametric EBLUP described by Opsomer *et al.* (2008) can be written as a pseudo-linear estimator, there is scope for an investigation of the performance of the Robust MSE estimator in this situation as well, and in particular a comparison with the computationally intensive bootstrap MSE estimator proposed by these authors.

The extension of the robust MSE approach to non-linear small area estimation situations remains to be done. However, since this approach is closely linked to robust population level MSE estimation based on Taylor series linearisation (as well as jackknife estimation of MSE, see Valliant, Dorfman and Royall, 2000, section 5.4.2), it should be possible to develop appropriate extensions for corresponding small area non-linear estimation methods. Although the relevant results are not provided here, some evidence for this is that the robust MSE estimation method described in section 2.1 has already been used to estimate the MSE of the MBDE when it is applied to variables that do not lend themselves to linear mixed modelling, e.g. those with a high proportion of zero values. See Chandra and Chambers (2009). More recently, the approach has also been successfully used to estimate the MSE of geographically weighted M-quantile small area estimators in situations where the small area values are spatially correlated (Salvati *et al.*, 2007). As noted earlier, it is of also of interest to examine whether this approach to MSE estimation can be used with predictors based on non-parametric small area models (Opsomer *et al.*, 2008) or with estimators based on outlier robust mixed effects models where the development of Prasad-Rao type MSE estimators is more difficult. This work will be reported elsewhere.

As is clear from the development in this paper, our preferred approach to MSE estimation assumes that the MSE of real interest is that defined by the area-specific model (1). This is in contrast to the usual approach to defining MSE in small area estimation, which adopts an area-averaged MSE concept as the appropriate measure of the accuracy of a small area estimator. As pointed out by Longford (2007), the ultimate aim in small area estimation is to make inferences about small area characteristics conditional on the realised (but unknown) values of small area effects, i.e. with respect to (1). One can consider this to be a design-based objective (as in Longford, 2007), or, as we prefer, a model-based objective that does not quite fit into the usual random effects framework for small area estimation. In either case we are interested in variability that is with respect to fixed area-specific expected values. This is consistent with the concept of variability that is typically applied in population level inference. As our simulations demonstrate this allows our MSE estimator to perform well from both a model-based (area effects vary between simulations) as well as a design-based (area effects fixed between simulations) perspective.

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REFERENCES

Chambers, R.L. and Dunstan, R. (1986). Estimating distribution functions from survey data. *Biometrika*, **73**, 597 - 604.

Chambers, R. and Tzavidis, N. (2006). M-quantile models for small area estimation. *Biometrika*, **93,** 255-268.

Chandra, H. and Chambers, R. (2009). Multipurpose weighting for small area estimation. *Journal of Official Statistics*, **25**, 1 – 18.

Henderson, C.R. (1953). Estimation of variance and covariance components. *Biometrics*, **9**, 226-252.

Jiang, J. and Lahiri, P. (2006). Estimation of finite population domain means: A model-assisted empirical best prediction approach. *Journal of the American Statistical Association*, **101**, 301–311.

Longford, N. T. (2007). On standard errors of model-based small-area estimators. *Survey Methodology*, **33**, 69-79.

Opsomer, J. D., G. Claeskens, M. G. Ranalli, G. Kauermann, and F. J. Breidt (2008). Nonparametric small area estimation using penalized spline regression. *Journal of the Royal Statistical Society, Series B*, **70**, 265-286.

Prasad, N.G.N and Rao, J.N.K. (1990). The estimation of the mean squared error of small area estimators. *Journal of the American Statistical Association*, **85**, 163-171.

Prasad, N.G.N. and Rao, J.N.K. (1999). On robust small area estimation using a simple random effects model. *Survey Methodology*, **25**, 67-72.

Rao, J.N.K. (2003). Small Area Estimation. Wiley, New York.

Royall, R.M. (1976). The linear least squares prediction approach to two-stage sampling. *Journal of the American Statistical Association*, **71**, 657 - 664.

Royall, R.M. and Cumberland, W.G. (1978). Variance estimation in finite population sampling. *Journal of the American Statistical Association*, **73**, 351 - 358.

Salvati, N., Tzavidis, N., Chambers, R. and Pratesi, M. (2007). Small area estimation via M-quantile geographical weighted regression. *Paper submitted for publication. A working paper is available from the authors upon request.*

Street, J.O, Carroll, R.J. and Ruppert, D. (1988). A note on computing robust regression estimates via iteratively reweighted least squares. *The American Statistician*, **42**, 152-154.

Tzavidis, N., Marchetti, S. and Chambers, R. (2009). Robust prediction of small area means and distributions. To appear in *Australian and New Zealand Journal of Statistics*.

Valliant, R., Dorfman, A.H. and Royall, R.M. (2000). Finite Population Sampling and Inference. Wiley, New York.

You, Y. and Rao, J.N.K. (2002). A pseudo-empirical best linear unbiased prediction approach to small area estimation using survey weights. *Canadian Journal of Statistics*, **30**, 431-439.

Table 1 Definitions of robust MSE estimators for different weighting methods.

Weighting Method	Definition of $\hat{\mu}_j, j \in i$	MSE Estimator
EBLUP (13)	(14)	(9)
MBDE (16)	(14)	(9)
M-quantile (20)	$x_j^T \hat{\beta}(\hat{q}_i)$	(8) with $\hat{\lambda}_j = 1$
Synthetic EBLUP (21)	(14)	(22) + (23)

Table 2 Parameter values used in model-based simulations.

Simulation	Type	σ_u^2	σ_e^2	$\rho = \sigma_u^2(\sigma_u^2 + \sigma_e^2)^{-1}$
SIM1-A	Gaussian	10.40	94.09	0.1
SIM1-B	Gaussian	40.32	94.09	0.3
SIM2-A	Chi-square	2.0	10.0	0.1667
SIM2-B	Chi-square	4.0	10.0	0.2857
SIM3-A	Mixture	10.40, 225.0	94.09	0.1, 0.7051
SIM3-B	Mixture	40.32, 225.0	94.09	0.3, 0.7051

Table 3 Median relative biases *RB(m)* and median relative root mean squared errors *RRMSE(m)* of estimators of small area means in model-based simulations. Note that results for SIM3-A and SIM3-B only refer to the 5 'outlier' areas.

Weighting Method	Simulation					
	SIM1-A	SIM1-B	SIM2-A	SIM2-B	SIM3-A	SIM3-B
	Average $n_i = 20$					
	RB(m)					
Direct	0.004	0.004	0.005	-0.024	0.001	0.001
EBLUP, (13)	0.005	0.006	0.004	-0.002	0.006	0.005
MBDE, (16)	0.006	0.006	0.005	-0.008	0.001	0.001
M-quantile, (20)	0.009	0.008	-0.002	0.002	-0.013	-0.013
	RRMSE(m)					
Direct	0.56	0.56	0.41	0.42	0.55	0.55
EBLUP, (13)	0.35	0.38	0.12	0.13	0.45	0.42
MBDE, (16)	0.55	0.55	0.41	0.43	0.55	0.55
M-quantile, (20)	0.41	0.41	0.13	0.13	0.36	0.36
	Average $n_i = 10$					
	RB(m)					
Direct	0.004	0.004	-0.001	0.003	-0.003	-0.003
EBLUP, (13)	-0.001	-0.002	0.000	0.000	0.000	0.003
MBDE, (16)	0.000	0.000	-0.002	0.000	-0.006	-0.006
M-quantile, (20)	-0.004	-0.004	0.0009	0.0004	-0.0036	-0.0045
	RRMSE(m)					
Direct	0.80	0.80	0.59	0.59	0.79	0.79
EBLUP, (13)	0.44	0.52	0.15	0.17	0.67	0.60
MBDE, (16)	0.80	0.80	0.59	0.58	0.78	0.78
M-quantile, (20)	0.57	0.57	0.18	0.18	0.58	0.57
	Average $n_i = 5$					
	RB(m)					
Direct	0.007	0.007	-0.005	-0.003	0.005	0.005
EBLUP, (13)	0.001	0.005	-0.002	0.003	0.008	0.011
MBDE, (16)	-0.002	-0.002	-0.005	0.004	-0.002	-0.002
M-quantile, (20)	-0.001	-0.001	-0.001	0.001	0.014	0.014
	RRMSE(m)					
Direct	1.13	1.13	0.84	0.84	1.13	1.13
EBLUP, (13)	0.53	0.69	0.19	0.22	1.00	0.87
MBDE, (16)	1.13	1.13	0.83	0.83	1.13	1.13
M-quantile, (20)	0.81	0.81	0.26	0.26	0.80	0.80

Table 4 Median relative biases *RB(M)* for MSE estimators in model-based simulations. Note that results for SIM3-A and SIM3-B only refer to the 5 'outlier' areas.

Weighting Method	MSE Estimator	Simulation					
		SIM1-A	SIM1-B	SIM2-A	SIM2-B	SIM3-A	SIM3-B
		Average $n_i = 20$					
Direct	SRS	0.11	0.11	-0.22	0.12	1.16	1.16
EBLUP, (13)	PR0	-0.83	-0.72	0.56	1.16	-15.65	-6.51
	PR1	-0.97	-0.72	0.64	1.08	-13.70	-5.81
	PR2	-0.92	-0.72	0.64	1.16	-14.65	-6.19
	Robust	3.89	-0.89	3.06	0.93	-2.56	-1.59
MBDE, (16)	Robust	-0.81	-0.80	-0.06	-0.42	-0.98	-0.98
M-quantile, (20)	Robust	-3.10	-1.66	-0.09	-1.90	11.26	11.04
		Average $n_i = 10$					
Direct	SRS	-0.09	-0.09	-0.38	-0.34	1.88	1.88
EBLUP, (13)	PR0	0.65	0.56	0.25	0.17	-21.99	-10.22
	PR1	0.47	0.56	0.00	0.17	-18.79	-8.91
	PR2	0.56	0.56	0.12	0.17	-20.39	-9.56
	Robust	15.36	1.72	8.94	2.36	-1.35	-1.16
MBDE, (16)	Robust	-0.73	-0.75	-0.38	-0.22	-0.92	-0.94
M-quantile, (20)	Robust	-2.65	-0.99	-1.73	2.00	6.50	4.50
		Average $n_i = 5$					
Direct	SRS	0.27	0.27	0.00	-0.90	-0.28	-0.28
EBLUP, (13)	PR0	3.51	-0.20	2.42	1.19	-30.64	-15.92
	PR1	3.04	-0.50	2.13	1.00	-25.77	-13.62
	PR2	3.16	-0.31	2.31	1.11	-28.16	-14.77
	Robust	37.52	4.38	24.11	8.93	-0.66	-0.68
MBDE, (16)	Robust	-0.24	-0.21	0.02	-0.09	1.29	1.24
M-quantile, (20)	Robust	-7.60	-6.17	5.70	5.00	5.89	3.60

Table 5 Median relative root mean squared errors *RRMSE(M)* for MSE estimators in model-based simulations. Note that results for SIM3-A and SIM3-B only refer to the 5 'outlier' areas.

Weighting Method	MSE Estimator	Simulation					
		SIM1-A	SIM1-B	SIM2-A	SIM2-B	SIM3-A	SIM3-B
				Average $n_i = 20$			
Direct	SRS	34	34	36	36	35	35
EBLUP, (13)	PR0	12	7	15	10	29	14
	PR1	14	7	17	11	27	13
	PR2	12	7	16	10	28	13
	Robust	62	31	70	49	42	32
MBDE, (16)	Robust	70	70	126	128	67	67
M-quantile, (20)	Robust	32	34	49	48	48	48
				Average $n_i = 10$			
Direct	SRS	49	49	52	52	50	50
EBLUP, (13)	PR0	19	10	23	15	40	21
	PR1	26	11	27	17	36	19
	PR2	21	10	24	15	38	20
	Robust	123	50	115	74	65	48
MBDE, (16)	Robust	74	74	128	129	75	75
M-quantile, (20)	Robust	44	46	68	68	62	59
				Average $n_i = 5$			
Direct	SRS	72	72	78	77	72	72
EBLUP, (23)	PR0	31	14	33	22	53	31
	PR1	48	18	44	28	48	29
	PR2	36	15	36	24	50	29
	Robust	234	81	193	121	86	70
MBDE, (24)	Robust	79	79	133	129	83	83
M-quantile, (28)	Robust	62	63	90	97	122	102

Table 6 Performances of estimators of regional means – Albanian household population

Weighting Method	Median n_i = 56		Median n_i = 11		Median n_i = 9	
	RB(m)	*RRMSE(m)*	*RB(m)*	*RRMSE(m)*	*RB(m)*	*RRMSE(m)*
Direct	0.03	6.15	-0.07	13.27	-0.09	16.64
EBLUP, (13)	0.42	5.90	1.41	11.61	1.62	12.42
MBDE, (16)	0.03	6.14	0.33	14.26	0.04	16.92
M-quantile, (20)	0.04	6.07	-0.09	13.44	-0.05	16.60

Table 7 Performances of MSE estimators – Albanian household population

Weighting Method/MSE Estimator	Median n_i = 56		Median n_i = 11		Median n_i = 9	
	RB(M)	*RRMSE(M)*	*RB(M)*	*RRMSE(M)*	*RB(M)*	*RRMSE(M)*
Direct/SRS	0.9	25	-0.3	57	-0.3	72
EBLUP/PR0	14.6	44	14.0	43	10.5	50
EBLUP/PR1	14.4	43	12.8	42	8.8	48
EBLUP/PR2	14.5	43	13.4	43	9.7	48
EBLUP/Robust	0.1	24	4.0	64	7.7	99
MBDE/Robust	-0.8	25	-3.6	54	-5.5	64
M-quantile/Robust	2.9	27	0.2	60	-2.0	75

Table 8 Performances of estimators of regional means – AAGIS farm population

Weighting Method	Median n_i = 55		Median n_i = 12		Median n_i = 8	
	RB(m)	*RRMSE(m)*	*RB(m)*	*RRMSE(m)*	*RB(m)*	*RRMSE(m)*
Direct	0.00	14.18	0.17	32.16	0.10	37.04
EBLUP, (13)	1.60	15.90	1.05	25.00	1.12	31.76
MBDE, (16)	-0.82	14.45	-1.76	31.06	-1.16	37.86
M-quantile, (20)	-0.03	11.76	-0.04	25.14	-0.23	32.63

Table 9 Performances of MSE estimators – AAGIS farm population

Weighting Method/MSE Estimator	Median n_i = 55		Median n_i = 12		Median n_i = 8	
	RB(M)	*RRMSE(M)*	*RB(M)*	*RRMSE(M)*	*RB(M)*	*RRMSE(M)*
Direct/SRS	0.3	64	0.4	126	0.7	169
EBLUP/PR0	23.7	209	22.6	555	17.7	701
EBLUP/PR1	24.9	190	16.6	406	19.0	597
EBLUP/PR2	22.3	221	23.3	483	31.1	782
EBLUP/Robust	11.5	157	14.5	253	17.8	261
MBDE/Robust	-0.8	190	1.4	178	1.3	364
M-quantile/Robust	-1.6	70	-1.0	154	-2.2	213

Table 10 Performance of EBLUP when there are zero-sample areas

Weighting Method	SIM1-A, average n_i = 10		AAGIS, median n_i = 8	
	RB(m)	*RRMSE(m)*	*RB(m)*	*RRMSE(m)*
Areas with n_i > 0: (13)	0.00	0.52	1.40	25.03
Areas with n_i = 0: (21)	-0.05	1.25	87.48	96.48

Table 11 Performances of MSE estimators for EBLUP when there are zero-sample areas

Weighting Method/MSE Estimator	SIM1-A, average n_i = 10		AAGIS, median n_i = 8	
	RB(m)	*RRMSE(m)*	*RB(m)*	*RRMSE(m)*
	Areas with n_i > 0			
(13)/PR0	0.5	11	33.5	939
(13)/Robust	0.7	50	28.1	318
	Areas with n_i = 0			
(21)/PR0	-1.8	35	-25.8	594
(21)/Robust	-3.6	34	-31.3	101

Table 12 Performances of estimators of regional means as number of simulations increases – AAGIS farm population (median n_i =12)

Weighting Method	Number of simulations					
	250	500	1000	250	500	1000
	RB(m)			*RRMSE(m)*		
Direct	-0.16	-0.11	0.17	31.75	31.06	32.16
EBLUP, (13)	0.64	1.50	1.05	26.32	25.12	25.00
MBDE, (16)	-2.17	-2.72	-1.76	30.98	31.27	31.06
M-quantile, (20)	0.91	0.04	-0.04	25.42	24.53	25.14

Table 13 Performances of MSE estimators as number of simulations increases – AAGIS farm population (median n_i =12)

Weighting Method/MSE Estimator	Number of simulations					
	250	500	1000	250	500	1000
	RB(M)			*RRMSE(M)*		
Direct/SRS	0.8	0.1	0.4	125	125	126
EBLUP/PR0	27.3	28.1	22.6	578	578	555
EBLUP/PR1	21.1	22.3	16.6	405	405	406
EBLUP/PR2	25.8	25.6	23.3	504	504	483
EBLUP/Robust	15.4	15.6	14.5	283	283	253
MBDE/Robust	1.9	4.6	1.4	251	251	178
M-quantile/Robust	-0.7	-0.8	-1.0	139	139	154

Figure 1 Area specific values of true RMSE (solid line) and average estimated RMSE (dashed line) obtained in the mixture-based simulations SIM3-A. Values for the PR0 estimator are indicated by Δ while those for the Robust estimator are indicated by o. Plots show results for the EBLUP (top), MBDE (centre) and M-quantile (bottom) estimators. Vertical line separates areas 26-30 with 'outlier' effects from 'well-behaved' areas 1-25.

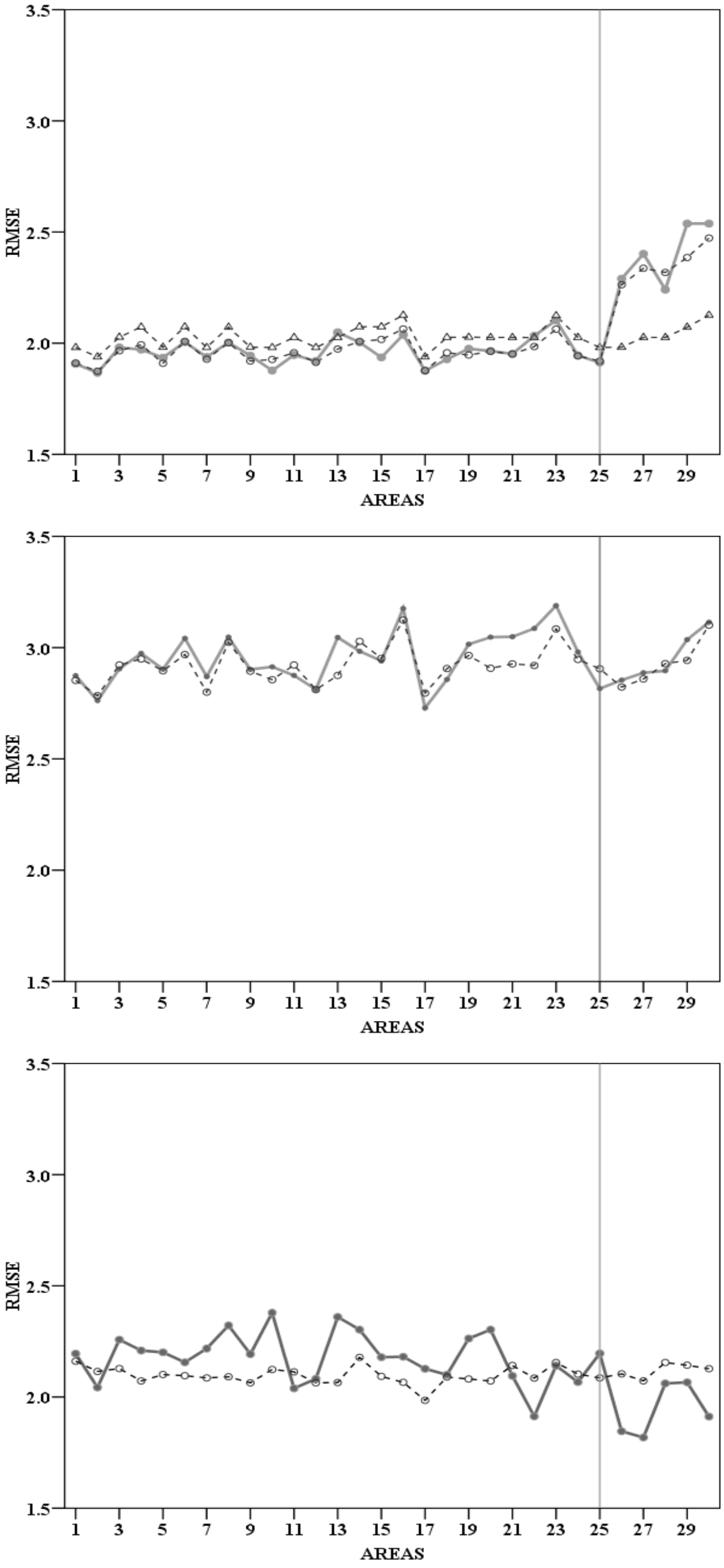

Figure 2 District level values of true design-based RMSE (solid line) and average estimated RMSE (dashed line) obtained in the design-based simulations using the Albanian household population. Districts are ordered in terms of increasing population size. Values for the PR0 estimator are indicated by Δ while those for the Robust estimator are indicated by o. Plots show results for the EBLUP (top), MBDE (centre) and M-quantile (bottom) estimators.

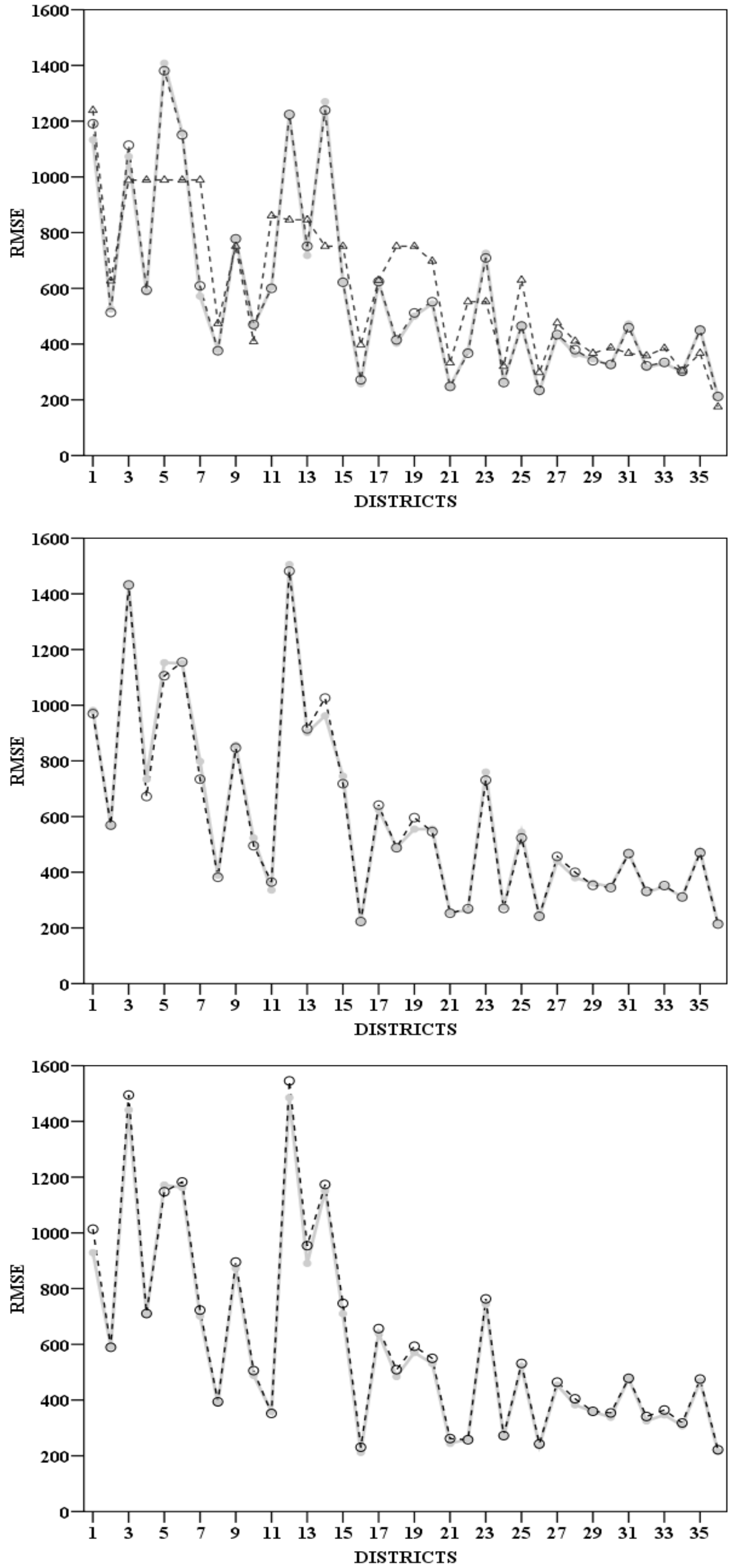

Figure 3 Regional values of true design-based RMSE (solid line) and average estimated RMSE (dashed line) obtained in the design-based simulations using the AAGIS farm population. Regions are ordered in terms of increasing population size. Values for the PR0 estimator are indicated by Δ while those for the Robust estimator are indicated by o. Plots show results for the EBLUP (top), MBDE (centre) and M-quantile (bottom) estimators.