HSC Mathematics - Extension 1

Workshop 4

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Estimation of roots

Halving the interval

Suppose x_1 and x_2 are two values of x such that $f(x_1)$ and $f(x_2)$ are of different sign. Then the solution to $f(x) = 0$ must lie in the interval (x_1, x_2). Then find the sign of $f(\frac{x_1+x_2}{2})$ to determine which half of the interval the root must lie. Repeat the process until convergence is achieved.

Newton's Method

Here we take an initial estimate of the root x_0 and calculate the gradient of the tangent to the curve $f(x)$ at $x = x_0$, $f'(x_0)$. This curve intersects the x-axis at x_1 which should be closer to the root than x_0.

The gradient of the curve can also be calculated from $\frac{f(x_0)}{x_0 - x_1}$

Therefore

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

Which gives on rearrangement

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{1}$$

1. **Exercise:**
   (a) Use two steps of halving the interval to find an approximation to $\sqrt[3]{40}$ using the starting interval $(3, 3.5)$.
   (b) Use two steps of Newton's Method to find an approximation to $\sqrt[3]{40}$ using the starting value $x_0 = 3$.
   (c) In what situations does Newton's Method fail?

Inverse Functions

A function has an inverse if it is monotonic or restricted to a domain where it is monotonic.
Rule for obtaining the inverse - swap x and y and make y the subject.

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2. **Exercise:** **(a)** Find the domain for which $f(x) = x^2 + 4x - 1$ is monotonic increasing.
(b) Find the inverse function, $f^{-1}(x)$, for this domain.

The domain of $f(x)$ is the range of $f^{-1}(x)$ and the range of $f(x)$ is the domain of $f^{-1}(x)$.

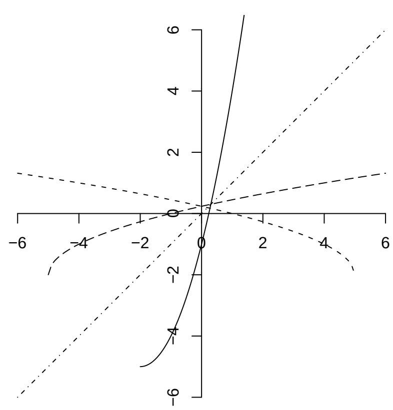
3. **Exercise:** Determine the domain and range of $f^{-1}(x)$ found in Exercise 2. Check that they are respectively the range and domain of $f(x)$.

Sketching a function and its inverse

The inverse function is the reflection of the original function in the line $y = x$.
Sometimes this is not easy to do. A simple method which works in all cases is:

- Use a sheet of plastic and a marking pen to trace $f(x)$ and the axes
- Rotate the plastic 90° anticlockwise so that the +ve y-axis becomes the -ve x-axis
- Flip the plastic 180°
- Copy the function from the plastic to the original plot

The graph shows $f(x) = x^2 + 4x - 1$, the plot when rotated 90° anticlockwise, the resulting plot of $f^{-1}(x)$ when flipped over and the line $y = x$.



4. **Exercise:**
   Consider the function $f(x) = \frac{1}{x^2+1}$
   (a) Sketch the function.
   (b) For what domain including 0 is the function monotonic decreasing?
   (c) Sketch the inverse function f^{-1} for the domain in (b).

   Note in the previous graph that $f(x)$ intersected with $f^{-1}(x)$ and the point of intersection lay on the line $y = x$.

   This is always true:
   if a function and its inverse intersect they do so on the line $y = x$.

5. **Exercise:** Find the coordinates of the point of intersection of $f(x)$ from Exercise 2 and its inverse. Check your result.

   Relationship between the derivatives of $f(x)$ and $f^{-1}(x)$

   Let (x_0, y_0) be a point on the curve $y = f(x)$.

   The derivative of the inverse function at y_0 is the reciprocal of the derivative of the function at x_0:

$$(f^{-1})' \ at \ \ y_0) = \frac{1}{f\,' \ at \ \ x_0}$$

   Or, given $y = f(x)$ then $x = f^{-1}(y)$ and:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \tag{2}$$

6. **Exercise:**
   Given $f(x) = \frac{x}{x+1}$ for $x > 1$, find the derivative of the inverse function
   (a) using Equation (2)
   (b) by finding the inverse function $x = f^{-1}(y)$, finding $\frac{dx}{dy}$ and expressing in terms of x.

   Composition of a function and its inverse

   We know that $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$. This is because the square and square root functions are inverses - applying one then the other yields the original value.

In the same way

$$f^{-1}[f(x)] = x \tag{3}$$

and

$$f[f^{-1}(x)] = x \tag{4}$$

7. **Exercise:**
Taking $f(x) = x^2 + 4x - 1$ from Exercise 2 and your definition of f^{-1}, choose a value of x in the domain and show that both $f^{-1}[f(x)]$ and $f[f^{-1}(x)]$ are equal to the chosen value.

Inverse Trigonometric Functions

The Inverse Trigonometric Functions are denoted by the power of -1, or the prefix *arc* or just *a*. For example the inverse *sin* function is denoted by $sin^{-1}x, arcsin\ x$ or $asin\ x$. Why are there different notations?

I shall use the power of -1 as it is more common at the school level.

8. **Exercise:**
Using the graphs of the trig. functions, restricted to a domain including 0 for which the functions are monotonic, sketch the inverse trig functions.

Note the domain and range of the inverse trig. functions.
How can we vary the domain and range?
Consider $y = \sin^{-1} x$ and generalize to $y = k\ \sin^{-1} f(x)$ where k is a constant.
The domain is determined from $-1 \leq f(x) \leq 1$.
The range is $-\frac{k\pi}{2} \leq y \leq \frac{k\pi}{2}$.

9. **Exercise:**
Find the domain and range of:
(a) $2\sin^{-1}(1-x)$
(b) $\frac{1}{3}\cos^{-1}\sqrt{x}$
(c) $4\tan^{-1} 3x$
(d) $-3\sin^{-1}(\ln x)$
(e) $-\frac{1}{2}\cos^{-1}(4x+1)$

The derivatives of the inverse trig. functions are:

$$\frac{d}{dx}\ \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \tag{5}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \tag{6}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \tag{7}$$

10. **Exercise:** Derive the formulae for the derivatives of (a) $sin^{-1}x$
    (b) $tan^{-1}x$

    To differentiate more complicated inverse trig. functions we need to use the function-of-a-function rule.

    Example: Differentiate $y = \sin^{-1}\sqrt{x}$

11. **Exercise:** Differentiate: (a) $\tan^{-1}\frac{1}{x}$
    (b) $2\cos^{-1}(x^2-1)$
    (c) $\sin^{-1}(\ln x)$
    (d) $4\cos^{-1}(e^x)$
    (e) $x\sin^{-1}x$
    (f) $\ln(\sin^{-1}x)$
    (g) $\sin^{-1}(\cos x)$
    (h) $\frac{\tan^{-1}\sqrt{x}}{x}$

Inverse Trig. Functions as primitives

The Table of Standard Integrals at the back of the exam paper gives:

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \tag{8}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} \tag{9}$$

Notice the similarity of

$$\int \frac{x}{a^2+x^2}dx$$

and

$$\int \frac{1}{a^2+x^2}dx$$

Both integrands are fractions with denominator to the power of 1. The numerator of the first integrand is related to the derivative of the denominator (thus the integral is a natural log) while the numerator of the second is a constant.

12. **Exercise:**

Some examples of integrals from past HSC papers:

Evaluate:
(a) $\int_0^1 \frac{1}{\sqrt{2-x^2}} dx$
(b) $\int \frac{1}{x^2+49} dx$
(c) $\int_{-1}^1 \frac{1}{\sqrt{4-x^2}} dx$
(d) $\int_0^{\sqrt{3}} \frac{4}{x^2+9} dx$

13. **Exercise:**

(a) Find the area enclosed by the function $f(x) = \frac{1}{\sqrt{4-x^2}}$, the x-axis and the lines $x = -1$ and $x = 1$.
(b) Find the volume of the solid of revolution formed when the function $y = \frac{1}{\sqrt{4+x^2}}$ is rotated around the x-axis from $x = 0$ to $x = 2$.
(c) Find the area bounded by the curve $y = \sin^{-1} x$, the x-axis and the line $x = \frac{1}{2}$.

An Inverse Trig. Identity

A property of $\sin^{-1} x$ and $\cos^{-1} x$ is

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \qquad (10)$$

14. **Exercise:** Prove this relationship in two different ways.