Extension 2 Conics

This topic mainly requires knowledge of:

- Co-ordinate Geometry
- Quadratic equations
  - Solving
  - Sums and Products of roots
  - Discriminant
- Lots of algebra and algebraic tricks
- Implicit Differentiation

Refer to the Syllabus (most particularly the Summary pages). Get it from the BOS website. [www.boardofstudies.nsw.edu.au](http://www.boardofstudies.nsw.edu.au) and follow the links to Syllabuses and then the relevant course. At this same website you can get feedback on previous HSC papers from the markers. Follow the links to the relevant exam papers and then to the Notes from the Marking Centre. There are good clues there, many of which I have used in this presentation.

The Summary outlines in dot point form all of the things that you can be tested on in each topic. There are four sections in Conics: Ellipses, Hyperbolae, Rectangular Hyperbolae and some General ideas.

The sections on the Ellipse, Hyperbolae and Rectangular Hyperbolae are all quite similar:

- A number of dot points on things that you should know
- A number of dot points of things that you should be able to find **(not memorise)**. Eg: find an equation of a tangent. Questions will either ask you to find it, which means to work it out, not just write it down, or they will give it to you to save time so that you can work on something a little more complex. In particular, you should learn the method for finding the Chord of Contact, which is not as straightforward as the others.
- A number of dot points on things that you should be able to prove. Learn these proofs, they’re in all good texts. Often they ask something connected to one of these proofs, although they are often asked in a not so obvious or alternative way, so it is good if you can recognise where they are heading. In 2009 they asked the Reflective property of an Ellipse proof. The question had 4 parts, was worth 8 marks and they led you through it step by step following the suggested method from the syllabus. Easy marks.
- For the Rectangular Hyperbola there are also a number of dot points regarding locus questions. These should only be asked in this section.
There are a number of common methods or tricks that need to be learned:

- When finding equations of tangents and normals you must recognise the need to divide all terms by \( a^2 b^2 \).

- When finding equations of tangents and normals you need to recognise that, since \( P(x_1, y_1) \) lies on the conic, when the term \( \frac{x_1^2}{a^2} \pm \frac{y_1^2}{b^2} \) arises it will satisfy the equation of the conic and, hence, have the value 1.

- Be prepared to use the definition of a conic \( PS = e \times PM \), or versions of it, whenever the requirement for \( PS \) arises. This is VERY common.

- Be prepared to substitute \( b^2 = a^2 (1 - e^2) \) or \( b^2 = a^2 (e^2 - 1) \) in order to simplify an expression. This is also VERY common.

- Remember that, for a rectangular hyperbola, \( e = \sqrt{2} \). We often get used to the idea that we don’t know \( e \) and are more used to finding ways to divide it out of our expressions and, so, often overlook this simple fact.

- For locus problems we can often show a fit to a type, especially if we are given a clue like “Show that the locus is a hyperbola.” In this case, if the question is difficult, we could just try to multiply the \( X \) and \( Y \) values and aim to show that the result is a constant giving us the \( XY = c^2 \) form. If this did not seem right then maybe \( X^2 - Y^2 \) could work for the alternate form of a rectangular hyperbola. If the question suggests that the locus is a straight line, often through the Origin, then it will satisfy \( Y = mX \) and we could try the \( Y \) value divided by the \( X \) value and again aim for a constant.
The point \( P(x_1, y_1) \) lies on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), where \( a > b > 0 \).

The equation of the normal to the ellipse at \( P \) is \( a^2 y_1 x - b^2 x_1 y = (a^2 - b^2) x_1 y_1 \).

(i) The normal at \( P \) passes through the point \( B(0, -b) \)

Show that \( y_1 = \frac{b^3}{a^2 - b^2} \) or \( y_1 = \pm b \).

(ii) Show that if \( y_1 = \frac{b^3}{a^2 - b^2} \), the eccentricity of the ellipse is at least \( \frac{1}{\sqrt{2}} \).
Solution to 2005 Q4c

i) The normal is given as $a^2y_x - b^2x_y = (a^2 - b^2)x_1y_1$, and substituting $B(0,-b)$, since we are told that the normal passes through this point, yields:

$$b^3x_1 = (a^2 - b^2)x_1y_1$$

Numerous methods here but we’re after $y_1$, so:

$$y_1 = \frac{b^3x_1}{(a^2 - b^2)x_1}$$

WATCH OUT: instinct suggests we now cancel the $x_1$’s which leads to the answer written in the paper as $y_1 = \frac{b^3}{(a^2 - b^2)}$ but then what?

According to the markers comments, a number of students did this. WHAT IF $x_1 = 0$ ???? We must ALWAYS ask ourselves this question when cancelling in this manner, it’s a common trick all the way down to 2 unit work. COULD $x_1 = 0$ ????

YES, it can, so we end up with the full answer by observing that IF $x_1 = 0$ $P$ will be on the $x$-axis and the $y_1$ value must be $\pm b$.

ii) On reading this part of the Q it seems like there is nowhere to go. We must ask ourselves what are the implications of $y_1 = \frac{b^3}{(a^2 - b^2)}$. As can be seen from the diagram, this means that $P$ is not on the $x$-axis and so $y_1 < b$ and this gives us our foothold. Again, many students missed this bit. Think hard, the clue is always there, it just takes time to see what they’re telling us.

Now, we have

$$\frac{b^3}{(a^2 - b^2)} < b$$

$$\frac{b^2}{(a^2 - b^2)} < 1 \quad \text{since } b > 0$$

$$b^2 < a^2 - b^2 \quad \text{since } a > b > 0$$

$$2b^2 < a^2$$
\[ \frac{b^2}{a^2} < \frac{1}{2} \]

\[ 1 - e^2 < \frac{1}{2} \quad \text{since } b^2 = a^2(1 - e^2) \]

\[ e^2 > \frac{1}{2} \]

\[ e > \frac{1}{\sqrt{2}} \quad \text{since } e > 0 \]
(d) The diagram shows the rectangular hyperbola $xy = c^2$, with $c > 0$.

The points $A(c, c)$, $R\left(ct, \frac{c}{t}\right)$ and $Q\left(-ct, -\frac{c}{t}\right)$ are points on the hyperbola, with $t \neq \pm 1$.

(i) The line $\ell_1$ is the line through $R$ perpendicular to $QA$.  
Show that the equation of $\ell_1$ is  

$$y = -tx + c\left(t^2 + \frac{1}{t}\right).$$

(ii) The line $\ell_2$ is the line through $Q$ perpendicular to $RA$.  
Write down the equation of $\ell_2$.

(iii) Let $P$ be the point of intersection of the lines $\ell_1$ and $\ell_2$.  
Show that $P$ is the point $\left(\frac{c}{t^2}, ct^2\right)$.

(iv) Give a geometric description of the locus of $P$.  

2

1

2

1
Solution to 2010 3d

A fairly standard, simple question with a nasty twist at the end.

i)

\[
m_{Q4} = \frac{c + c}{t} = \frac{c}{c + ct}
\]

\[
m_{Q4} = \frac{ct + c}{t} \times \frac{1}{c + ct}
\]

\[
m_{Q4} = \frac{1}{t}
\]

\[
\therefore m_{L1} = -t
\]

Equation L1: \[y - \frac{c}{t} = -t(x - ct)\]

\[y = -tx + ct^2 + \frac{c}{t}\]

\[y = -tx + c(t^2 + \frac{1}{t})\]

ii)

\[
m_{R4} = \frac{c - c}{t} = \frac{c}{c - ct}
\]

\[
m_{R4} = \frac{ct - c}{t} \times \frac{1}{c - ct}
\]

\[
m_{R4} = -\frac{1}{t}
\]

\[
\therefore m_{L2} = t
\]

Equation L2: \[y + \frac{c}{t} = t(x + ct)\]

\[y = tx + ct^2 - \frac{c}{t}\]

\[y = tx + c(t^2 - \frac{1}{t})\]
iii) At P:

\[-tx + ct^2 + \frac{c}{t} = tx + ct^2 - \frac{c}{t}\]

\[\frac{2c}{t} = 2tx\]

\[x = \frac{c}{t^2}\]

\[\therefore y = -t \times \frac{c}{t^2} + ct^2 + \frac{c}{t}\]

\[y = ct^2\]

\[\therefore P \text{ is } \left(\frac{c}{t^2}, ct^2\right)\]

iv) \[x = \frac{c}{t^2}, \quad y = ct^2\]

\[x \times y = \frac{c}{t^2} \times ct^2\]

\[xy = c^2\]

Beauty!!! It’s just the original hyperbola!!!

BUT this is too easy. It always pays to be a bit suspicious.

Check your diagram for any clues and note that, since \(c > 0\) and P is \(\left(\frac{c}{t^2}, ct^2\right)\)

P can only have positive coordinates

ie. The locus is the rectangular hyperbola \(xy = c^2\) but only in the first quadrant.

Nasty really because it was only worth one mark and without the fine tuning you score zero. Markers’ comments indicated that very picked up the trick.
In the diagram the secant $PQ$ of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) meets the directrix at $R$. Perpendiculars from $P$ and $Q$ to the directrix meet the directrix at $U$ and $V$ respectively. The focus of the ellipse which is nearer to $R$ is at $S$.

Copy or trace this diagram into your writing booklet.

(i) Prove that \( \frac{PR}{QR} = \frac{PU}{QV} \).  

(ii) Prove that \( \frac{PU}{QV} = \frac{PS}{QS} \).  

(iii) Let $\angle PQS = \phi$, $\angle RSQ = \theta$ and $\angle PRS = \alpha$.  

By considering the sine rule in $\Delta PRS$ and $\Delta QRS$, and applying the results of part (i) and part (ii), show that $\phi = \pi - 2\theta$.  

(iv) Let $Q$ approach $P$ along the circumference of the ellipse, so that $\phi \to 0$.  

What is the limiting value of $\theta$?
Solution to 2007 Q7b

i) Obtained very easily from similar triangles

Need to do more than just write down the answer. You must convince the markers that you know what you’re talking about. So make sure that you state that you’re getting the result by comparing corresponding sides of similar triangles or another good reason.

ii) This is the use of the \( PS = e \times PM \) result.

Where

\[
\frac{PS}{QS} = \frac{ePU}{eQV}
\]

\[
\frac{PS}{QS} = \frac{PU}{QV}
\]

Note that \( \frac{PU}{QV} \) appears in both i) and ii) in different forms, so at this stage we must be thinking that we will use these results later.

iii) Do as they say, use the sine rule, first mark is easy!!!!

In \( \triangle PRS \)

\[
\frac{\sin(\phi + \theta)}{PR} = \frac{\sin \alpha}{PS}
\]

In \( \triangle QRS \)

\[
\frac{\sin \theta}{QR} = \frac{\sin \alpha}{QS}
\]

\[
\frac{\sin(\phi + \theta)}{\sin \alpha} = \frac{PR}{PS}
\]

And from i) and ii)

\[
\frac{PR}{QR} = \frac{PS}{QS}
\]

Or

\[
\frac{PR}{PS} = \frac{QR}{QS}
\]
So,
\[
\frac{\sin(\phi + \theta)}{\sin \alpha} = \frac{\sin \theta}{\sin \alpha}
\]

\[
\sin(\phi + \theta) = \sin \theta \quad \text{since} \quad \alpha \neq 0, \pi
\]

\[
\therefore \quad \phi + \theta = \pi - \theta \quad \text{this step missed by many}
\]

\[
\phi = \pi - 2\theta
\]

iv) from above \quad \phi + 2\theta = \pi

and, as \quad \phi \to 0 \quad 2\theta \to \pi

\[
\theta \to \frac{\pi}{2}
\]

Note that this is just a question on the syllabus proof: the part of the tangent between the point of contact and the directrix subtends a right angle at the focus.

Try to look for clues like this in questions. Ask yourself what is going on when Q moves to P.
(b) The point $P(a \cos \theta, b \sin \theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.

The acute angle between $OP$ and the normal to the ellipse at $P$ is $\phi$.

(i) Show that $\tan \phi = \left(\frac{a^2 - b^2}{ab}\right) \sin \theta \cos \theta$.

(ii) Find a value of $\theta$ for which $\phi$ is a maximum.
Solution to 2014 Q14b

i) The big issue here is in deciding how to start. Looking carefully at the diagram and the information given we realise that we can use the angle between 2 lines formula.

Using implicit differentiation and standard methods we can get:

\[ m_{NORM} = \frac{a\sin\theta}{b\cos\theta} \quad \text{and} \quad m_{OP} = \frac{b\sin\theta}{a\cos\theta} \]

\[ \therefore \tan\phi = \frac{\frac{a\sin\theta}{b\cos\theta} - \frac{b\sin\theta}{a\cos\theta}}{1 + \frac{a\sin\theta}{b\cos\theta} \times \frac{b\sin\theta}{a\cos\theta}} \]

\[ \tan\phi = \frac{\frac{a^2\sin\theta - b^2\sin\theta}{abc\cos\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} \]

\[ \tan\phi = \frac{(a^2 - b^2)\sin\theta\cos\theta}{abc\cos\theta} \times \frac{\cos^2\theta}{1} \]

\[ \tan\phi = \frac{(a^2 - b^2)\sin\theta\cos\theta}{ab} \]

since \( a > b > 0 \) and \( \theta \) is in the first quadrant, \( \therefore \sin\theta, \cos\theta \) positive

ii) Again, the trick is in choosing which method to use. First instinct may be to differentiate to find stationary points but realising that \( 2\sin\theta\cos\theta = \sin2\theta \) gets us

\[ \tan\phi = \frac{(a^2 - b^2)\sin2\theta}{2ab} \]

And, because \( a, b \) are both constant, \( \tan\phi \) and, hence, \( \phi \), will be a maximum when \( \sin2\theta \) is a maximum.

This will occur when \( \sin2\theta = 1 \)

\[ \text{ie.} \quad 2\theta = \frac{\pi}{2} \]

\[ \theta = \frac{\pi}{4} \]

Remember, this is the eccentric angle of \( P' \) which is vertically above \( P \) on the auxiliary circle. It is not the angle between the \( x \) – axis and \( OP \).
The diagram shows an ellipse with eccentricity $e$ and foci $S$ and $S'$. The tangent at $P$ on the ellipse meets the directrices at $R$ and $W$. The perpendicular to the directrices through $P$ meets the directrices at $N$ and $M$ as shown. Both $\angle PSR$ and $\angle PS'W$ are right angles.

Let $\angle MPW = \angle NPR = \beta$.

(i) Show that

\[
\frac{PS}{PR} = e \cos \beta
\]

where $e$ is the eccentricity of the ellipse.

(ii) By also considering $\frac{PS'}{PW}$ show that $\angle RPS = \angle WPS'$.  

\[\text{\hfill 2} \]
**Solution to 2007 Q7c**

i) \( PS = e \times PN \) 

Use this result again which leads us to investigate \( PN \) and \( \Delta PNR \)

\[
\cos \beta = \frac{PN}{PR}
\]

\( PN = PR \cos \beta \)

So, \( PS = e \times PR \cos \beta \)

\[
\frac{PS}{PR} = e \cos \beta
\]

ii) Follow their instruction and using the above method yields \( \frac{PS'}{PW} = e \cos \beta \)

and so, \( \frac{PS}{PR} = \frac{PS'}{PW} \)

and, by considering \( \Delta PRS \) and \( \Delta PWS' \)

\[
\cos(\angle RPS) = \cos(\angle WPS')
\]

\[\therefore \angle RPS = \angle WPS' \quad \text{since both angles are acute}\]

This time we have proven the reflective property of the ellipse without realising it. It states that the tangent to an ellipse at a point on it is equally inclined to the focal chords through the point.

Note also that these two parts, 7b and 7c were quite easy really and were worth a total of 9 marks. Make sure that you use your reading time to identify questions late in the paper that you think that you can do. Avoid earlier questions that stump you if you know there are others later that you will do better on. Read the paper backwards in reading time for this purpose. Start at 16, then 15 etc. (Note: old papers had only 8 Q’s)
The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The tangents at $P$ and $Q$ meet at $T$.

(i) Show that the equation of the tangent at $P$ is $\frac{x_1}{a^2} x + \frac{y_1}{b^2} y = 1$.  

(ii) Show that $T$ lies on the line $\frac{(x_1 - x_2)}{a^2} x + \frac{(y_1 - y_2)}{b^2} y = 0$.  

(iii) Let $M$ be the midpoint of $PQ$. Show that $O, M$ and $T$ are collinear.
Solution to 2008 Q4b

i) Assume result, tangent at $P$ is \( \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \)

ii) Similarly, tangent at $Q$ is \( \frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1 \)

solving simultaneously for $T$:

\[
\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_2x}{a^2} + \frac{y_2y}{b^2}
\]

\[
\frac{x}{a^2}(x_1 - x_2) + \frac{y}{b^2}(y_1 - y_2) = 0
\]

And so $T$ must lie on this line as its coordinates are the solution to the equation and so satisfy the equation. Note that there is no need to find the actual coordinates.

iii) This part was tricky and the Markers’ comments specifically mention that students got very bogged down trying to find the coordinates of $T$.

We have to ask ourselves what are they getting at? What was the point of part ii)? They have given us a line and told us that $T$ is on it. We need to realise that this is probably the line that they are all on and check.

It is easy to see that $O$ will be on the line so the line is the equation of $OT$. Now we only need to find the coordinates of $M$ and check that it lies on the line that we already have.

Coordinates of $M$ are easy: \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)

Substituting these coordinates into the LHS of the equation of $OT$:

\[
LHS = \frac{(x_1 + x_2)(x_1 - x_2)}{2a^2} + \frac{(y_1 + y_2)(y_1 - y_2)}{2b^2}
\]

\[
LHS = \frac{1}{2}(\frac{x_1^2 - x_2^2}{a^2} + \frac{y_1^2 - y_2^2}{b^2})
\]

\[
LHS = \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2}\right)
\]
\[ LHS = \frac{1}{2}(1 - 1) \]

\[ LHS = 0 = RHS \]

\[ \therefore M \text{ also lies on this line.} \]

Hence, \( O, M, T \) are collinear.

Note the 3 marks usually indicate 3 tasks:

- Show that \( O \) is on the line
- Find the coordinates of \( M \)
- Show that \( M \) lies on the line

Don’t give up if you don’t “get” the question. There are still marks available for what you can do.
HSC 2008 Question 6b

Let \( P(a \sec \theta, b \tan \theta) \) be a point on the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) where \( a > 0 \) and \( b > 0 \) as shown in the diagram. The foci of the hyperbola are \( S \) and \( S' \), and \( \ell \) is the tangent at the point \( P \).

The points \( R \) and \( R' \) lie on \( \ell \) so that \( SR \) and \( S'R' \) are perpendicular to \( \ell \).

(i) Show that the line \( \ell \) has equation

\[
bx \sec \theta - ay \tan \theta - ab = 0.
\]

(ii) Show that \( \frac{ab(e \sec \theta - 1)}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}} = SR \).

(iii) Show that \( SR \times S'R' = b^2 \).
**Solution to 2008 Q6b**

i) Assume this result

ii) Applying Perpendicular distance formula yields:

\[ SR = \frac{|ab(e\sec \theta - 1)|}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}} \]

which is the required answer except for the absolute value.

We must not just drop this absolute value to suit. The markers’ comments specifically stated that many students did not consider the situation properly. This will hurt in part iii) as you will see.

We need to consider whether \( ab(e\sec \theta - 1) \) is positive or negative.

Now, \( a, b \) are both positive, \( e > 1 \) and \( \theta \) is acute so \( \cos \theta < 1 \)

\[ \therefore \sec \theta > 1 \]

\[ e\sec \theta > 1 \]

\[ e\sec \theta - 1 > 0 \]

So, \( ab(e\sec \theta - 1) > 0 \) and it will be OK to take away the absolute value, giving the required answer.

iii) 3 marks here, so a bit of work. Obvious first step will be to use Perpendicular distance formula to give:

\[ S'R' = \frac{|-ab(e\sec \theta + 1)|}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}} \]

And if we don’t get it right with the absolute value here, the question won’t work out. From what we did above, it is obvious that \(-ab(\sec \theta + 1) < 0\) and so

\[ S'R' = \frac{ab(e\sec \theta + 1)}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}} \] when we drop the abs. val.

Which leads to

\[ SR \times S'R' = \frac{a^2 b^2 (e^2 \sec^2 \theta - 1)}{a^2 \tan^2 \theta + b^2 \sec^2 \theta} \]
Which looks like a mess but using the old favourite \( b^2 = a^2(e^2 - 1) \) in the denominator (why only in the denominator? Because we note the \( b^2 \) in the numerator and realise all else must eventually cancel out)

\[
SR \times S'R' = \frac{b^2(e^2 \sec^2 \theta - 1)}{\tan^2 \theta + (e^2 - 1)\sec^2 \theta}
\]

since \( a^2 \neq 0 \)

\[
SR \times S'R' = \frac{b^2(e^2 \sec^2 \theta - 1)}{\tan^2 \theta + e^2 \sec^2 \theta - \sec^2 \theta}
\]

\[
SR \times S'R' = \frac{b^2(e^2 \sec^2 \theta - 1)}{e^2 \sec^2 \theta - 1}
\]

since \( \tan^2 \theta - \sec^2 \theta = -1 \)

\[
SR \times S'R' = b^2
\]

Yeeehhar!!

It looked nasty but was really only a load of messy algebra, knowledge of the perpendicular distance formula, a use of the usual \( b^2 = a^2(e^2 - 1) \) trick and taking care with absolute value signs. Other than that, it was just follow their instructions as to what to work out.