Fractions

4. Fractions, Decimals, and Percentages

If you are unsure about what a fraction is, please refer to *Fractions – 1. Manipulating Fractions*.

INTRODUCTION

To relate *fractions* and *decimals*, let's think about our *decimal system of numbers* (or *place value*). This system gives us a very efficient way to write any number we please, just by using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in columns, where each column represents a different size.

Here are the headings for the columns that you are familiar with:

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones/Units

Each time we move a column from a column to one directly on its left, we have increased the place value by a factor of ten, i.e. we have multiplied by 10. So, for example, if we start with 3, i.e. we have a 3 in the units column, but move it one column to the left we end up with $30 = 3 \times 10$. Or if we start with 58, i.e. a 5 in the tens column and an 8 in the units column, but move them one column to the left we end up with $580 = 58 \times 10$. We have to add a zero digit onto our number because once we start our number all columns to the right must be filled. In the same way, moving one column from left to right is the same as if we divide by 10. So, for example, moving all the digits of 4930 one place to the right gives $493 = 4930 \div 10$.

$\times 10$ ← $\div 10$ →

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones/Units

All the columns listed so far refer to *whole numbers*. But what happens if we have a whole number where the last digit is not 0 and we want to divide it by 10? How can we pick the number up and move it across a column? We need to be able to show that we are no longer dealing with a whole number and will only have part, or a *fraction*, of a whole. This is why we have a *decimal point*. The decimal point separates whole numbers from parts of a whole:

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones/Units	.

Let's begin by dividing 1 by 10. We can write this as $1 \div 10$, but you probably realise you can also write this as $\frac{1}{10}$. Using the fact that when we divide by 10 we move our numbers across a column from left to right, we would have:

$\div 10$ →

Units/Ones	.	?
1 →		
0	.	1

So what do we call our new column? 0.1 (which some people write without the leading 0) must be the same as $\frac{1}{10}$ and so we call the new column *tenths*. (The "th" on the end tells us that we are not dealing with a whole number any more, but part of a number.) Since we know that 0.1 is the same as $\frac{1}{10}$, we write this mathematically as $0.1 = \frac{1}{10}$.

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Our columns continue on the right, for example, if we divide each tenth into ten bits (that is we calculate $\frac{1}{10} \div 10$) we would get on hundred bits and so we call the next column across *hundredths*. So, $\frac{1}{10} \div 10 = \frac{1}{100}$, or $0.1 \div 10 = 0.01$. This is what we have if we put it all together in our decimal system:

Thousands	Hundreds	Tens	Ones/Units	.	Tenths	Hundredths	Thousandths

Where the columns continue indefinitely on both sides of the table.

Therefore we are now able to easily move between decimals and fractions whose denominator is a power of ten. That is we know:

$\frac{1}{10} = 0.1$, $\frac{1}{100} = 0.01$, and $\frac{1}{1000} = 0.001$,

but we also have:

$\frac{3}{10} = 0.3$, $\frac{3}{100} = 0.03$, $\frac{23}{100} = 0.23$, $\frac{23}{1000} = 0.023$, $\frac{594}{1000} = 0.594$, and $\frac{1234}{1000} = 1.234$.

Remember, we can only put one digit in each column of our decimal system. The number of 0s in the denominator tells us which column the last digit of the numerator goes into.

Here is an exercise for you to try, you can check your answer with the solution at the end of the resource.

EXERCISES

1. Match the decimals and fractions in the following table.

Fraction	Decimal
$\frac{41}{100}$	0.41
$\frac{41}{1000}$	4.1
$\frac{12}{1000}$	0.12
$\frac{41}{10}$	0.041
$\frac{12}{100}$	0.012

MOVING FROM FRACTIONS TO DECIMALS

We have just looked at the relationship between fractions and decimals, as long as the denominator of the fraction is 10, 100, 1000, etc. But what about fractions such as $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and so on? In these cases where the denominator is not a power of 10, we need to consider a different method to work out their decimal equivalents.

For this we go back to the definition of a fraction, where the top number is divided by the bottom number (note we did use this when working out $\frac{1}{10}$ in decimals). For $\frac{1}{2}$, we need to calculate $1 \div 2$, so we set up our division:

$$2\overline{)1}$$

We see that 2 will not divide into 1, but we know that 1 is the same as 1.0, so our division becomes:

$$2\overline{)1.0}$$

There are no 2s in 1, so we put a 0 above the 1, followed by a decimal point above the decimal point in the 1.0:

$$\begin{array}{r} 0. \\ 2\,\overline{)\,1.0} \end{array}$$

Now we ask how many 2s in 10? There are 5, so we put 5 above the line:

$$\begin{array}{r} 0.5 \\ 2\,\overline{)\,1.0} \end{array}$$

There is no remainder, so we stop and obtain $\frac{1}{2} = 0.5$.

For $\frac{2}{3}$ we calculate $2 \div 3$,

$$3\,\overline{)\,2}$$

Again 3 will not divide into 2 so we put a 0 above the 2, and attach a decimal point and a 0 after the 2:

$$\begin{array}{r} 0. \\ 3\,\overline{)\,2.0} \end{array}$$

Now we ask how many 3s there are in 20? There are 6, so:

$$\begin{array}{r} 0.6 \\ 3\,\overline{)\,2.0} \end{array}$$

But there is also 2 left over, so we need to add another zero to the end (we can add as many zeros as we like after a decimal point as it doesn't change the number, i.e. $2.00000 = 2$) and carry the 2 over:

$$\begin{array}{r} 0.6\phantom{^{2}0} \\ 3\,\overline{)\,2.0^{2}0} \end{array}$$

And again we ask how many 3s in 20? This gives us 6 with 2 remainder again.

$$\begin{array}{r} 0.6\phantom{^{2}}6\phantom{^{2}0} \\ 3\,\overline{)\,2.0^{2}0^{2}0} \end{array}$$

We can see that this pattern will continue indefinitely and we will obtain 0.666666666 .... When we have a repeated number we usually write this with a dot placed over that number, that is $0.6666666\ldots = 0.\dot{6}$. If the repeating pattern has two or more numbers we place a line over the repeating segment e.g. $0.109090909\ldots = 0.1\overline{09}$.

Thus, we have $\frac{2}{3} = 0.\dot{6}$.

This division method allows us to turn any fraction into a decimal. Here are some for you to try. You can check your answers with the solutions at the end of the resource.

EXERCISES

2. Match the decimals and fractions in the following table.

Fraction	Decimal
$\frac{4}{5}$	$0.1\dot{6}$
$\frac{1}{6}$	0.8
$\frac{3}{8}$	$0.\dot{2}$
$\frac{2}{9}$	$0.41\dot{6}$
$\frac{5}{12}$	0.375

We have found how to convert fractions into their decimal form, now we will look at how to work the other way. This is generally much easier. Let's say we want to know what the decimal 0.59 is as a fraction. From our decimal system above, we know that this number consists of 5 tenths and 9 hundredths and we can write it as $\frac{59}{100}$. We now check to see if this fraction can be simplified, it can't so we write $0.59 = \frac{59}{100}$.

Now, let's look at 0.155? We can write this as $\frac{155}{1000}$, and we now check to see if we can simplify. We note that both the numerator and denominator are divisible by 5 so we can simplify $\frac{155 \div 5}{1000 \div 5} = \frac{31}{200}$. We can't simplify anymore so we know $0.155 = \frac{31}{200}$.

Here are some more exercises for you to try. You can check your answers with the solutions at the end.

EXERCISES

Change the following decimals to fractions; write your answer in its simplest form.

3. 0.7 4. 0.89 5. 0.2 6. 0.1 7. 0.10

8. 0.123 9. 0.5 10. 0.24 11. 0.125 12. 1.4

Note: Converting decimals that have a repeating pattern to fractions is harder, so we will leave it for the moment and come back to it in the last section of the resource.

PERCENTAGES

Percentages relate to fractions and decimals because *per cent* means *out of* 100. So, for example, 80% means 80 out of 100, or $\frac{80}{100}$. Now that we have written it as a fraction we can both simplify it or use the previous methods to write it as a decimal. Firstly, let's simplify it:

$$\frac{80}{100} = \frac{80 \div 20}{100 \div 20} = \frac{4}{5}.$$

Now to write it as a decimal, we know from the place value system that $\frac{80}{100}$ is the same as 0.80, which is the same as 0.8.

So the relationship is $80\% = \frac{4}{5} = 0.8$. All these forms represent the same relatively high proportion. If you got 80%, 4 out of 5, or 0.8 for an assessment task, you should be pretty happy!

So, in summary, to change from a percentage to a fraction we write the percentage over 100 and simplify, while to change to a decimal we divide by 100, that is mover the decimal point two sports to the left.

Now let's change a decimal or fraction into a percentage.

It is easiest to change a decimal to a percentage as we just do the reverse of what we did above. That is we move the decimal point two spots to the right (which is equivalent to multiplying by 100). So, to change 0.5 to a percentage we just calculate $0.5 \times 100 = 50\%$.

Or to change 1.258 to a percentage we do the same $1.258 \times 100 = 125.8\%$. Does such a percentage make sense? Yes! Things can increase by 125.8%, it does not matter that the number is greater than 100.

Now let’s consider changing a fraction to a percentage. One way to do this is to use our method to convert it to a decimal then multiply by 100 (as we do when converting decimals to percentages). For example, if we want to write $\frac{1}{2}$ as a percentage we know for previous work that $\frac{1}{2} = 0.5$, and that $0.5 = 50\%$, so we end up with $\frac{1}{2} = 50\%$.

This also turns out to be the best way. Some people prefer to multiply by 100 first, then do the division but this does not change the amount of work done. E.g.

$$\begin{aligned} \frac{1}{2} \times 100 &= \frac{1}{2} \times \frac{100}{1} \\ &= \frac{100}{2} \\ &= 50\%. \end{aligned}$$

Let’s try a harder example, $\frac{3}{8}$. We can change this to a decimal by doing the division

$$\begin{array}{r} 0.3\ 7\ 5 \\ 8\,\overline{)\,3.\,0^{6}0^{4}0} \end{array}$$

And we now move the decimal point two spots to the right to get the percentage. So, $\frac{3}{8} = 0.375 = 37.5\%$. Of course we could have multiplied by 100 first to get $\frac{300}{8}$, then performed the division or simplified the fraction in order find out the percentage:

$$\begin{array}{r} 3\ \ 7.\ 5 \\ 8\,\overline{)\,30^{6}0.^{4}\,0} \end{array}$$

So 37.5%, or by simplifying the fraction:

$$\begin{aligned} \frac{300}{8} &= \frac{300 \div 4}{8 \div 4} \\ &= \frac{75}{2} \\ &= 37\frac{1}{2}. \end{aligned}$$

That is, $37\frac{1}{2}\%$.

Here are some conversions for you to try. You can check your answers with the solutions at the end of the resource.

EXERCISES

13. Complete the following table.

Fraction	Decimal	Percentage
$\frac{2}{5}$		
	0.23	
		75%
		30%
	0.10	
$\frac{2}{3}$		

RECURRING DECIMALS TO FRACTIONS

Now let's look at how to change decimals that have a repeating pattern in them to fractions. We can't simply use our place value system to write it as a fraction because there is no end to the number (so we have no denominator to use). Instead we have to be a little trickier.

Suppose we want to write $0.\dot{3}$ as a fraction. Remember this means $0.33333333\ldots$, our first aim is remove the neverendingness of this decimal. Unfortunately to do this we need to introduce a very small bit of algebra (please don't run away). We are going to let x stand for our number. That is, $x = 0.3333333\ldots$.

Our next step is to multiply x by 10, so $10x = 3.3333333\ldots$. Now we have two numbers that have the same repeating pattern of decimals! So we can subtract them and get rid of this pattern.

$$\begin{aligned} 10x - x &= 3.333333\ldots - 0.333333\ldots \\ &= 3. \end{aligned}$$

But $10x - x$ is the same as $9x$, so we have $9x = 3$ and $x = \frac{3}{9}$. We have written x as a fraction! So $0.\dot{3} = 0.333\ldots = \frac{3}{9}$. Now we can still simplify this fraction by dividing to and bottom by 3: $\frac{3\div 3}{9\div 3} = \frac{1}{3}$, so the final result is $0.\dot{3} = 0.33\ldots = \frac{1}{3}$.

But can we always get rid of the recurring part of a decimal like this? Yes! Let's consider a harder example such as $0.1\overline{09}$. We remember that $0.1\overline{09} = 0.1090909\ldots$ with the two digits 0 and 9 repeating forever. We again set

$$x = 0.10909\ldots$$

but now since there are two digits repeating we need to multiply x by 100 (in general the number of zeros in the multiplier is equal to the number of digits that repeat).

$$100x = 10.9090909\ldots$$

Now we subtract x from this to obtain

$$\begin{aligned} 99x &= 10.9090909\ldots - 0.1090909\ldots \\ &= 10.8 \end{aligned}$$

If you need a reminder on how to subtract decimals please see *Decimals – 1. Addition and Subtraction*. Now since the right hand side isn't a whole number we need to make it one by multiplying by 10, so

$$10 \times 99x = 108$$

Or

$$990x = 108$$

So, $= \frac{108}{990}$. We can then simplify this if necessary. For example both top and bottom are divisible by 9, so

$$\begin{aligned} x &= \frac{108 \div 9}{990 \div 9} \\ &= \frac{12}{110} \\ &= \frac{6}{55} \end{aligned}$$

Therefore $0.1\overline{09} = \frac{6}{55}$.

Note you could also multiply by 10 at the start to ensure that the decimal places only contain the repeating digits, $10x = 1.090909\ldots$, then multiply by 100, $1000x = 109.0909\ldots$, and subtract $10x$, $990x = 109.09\ldots - 1.09\ldots = 108$. This way it ensures you only need to worry about the whole number portion in the subtraction.

If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

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$\frac{1}{6}$	0.8
$\frac{3}{8}$	$0.\dot{2}$
$\frac{2}{9}$	$0.41\dot{6}$
$\frac{5}{12}$	0.375

3. $0.7 = \frac{7}{10}$

4. $0.89 = \frac{89}{100}$

5. $0.2 = \frac{2 \div 2}{10 \div 2} = \frac{1}{5}$

6. $0.1 = \frac{1}{10}$

7. $0.10 = \frac{10 \div 10}{100 \div 10} = \frac{1}{10}$

8. $0.123 = \frac{123}{1000}$

9. $0.5 = \frac{5 \div 5}{10 \div 5} = \frac{1}{2}$

10. $0.24 = \frac{24 \div 4}{100 \div 4} = \frac{6}{25}$

11. $0.125 = \frac{125 \div 25}{1000 \div 25} = \frac{5 \div 5}{40 \div 5} = \frac{1}{8}$

12. $1.4 = \frac{14 \div 2}{10 \div 2} = \frac{7}{5}$

13. Complete the following table.

Fraction	Decimal	Percentage
$\frac{2}{5}$	0.4	40%
$\frac{23}{100}$	0.23	23%
$\frac{75}{100}=\frac{3}{4}$	0.75	75%
$\frac{30}{100}=\frac{3}{10}$	0.3	30%
$\frac{1}{10}$	0.10	10%
$\frac{2}{3}$	$0.\dot{6}$	$66.\dot{6}\%$