

Calculations

Order of Operations in Mathematics

INTRODUCTION

Operations in mathematics are all those procedures like $+$, $-$, \times and \div , together with $(\)$, raising to a power and $\sqrt{\quad}$, which are used in calculating, and, especially, which are used to write down what to calculate.

Why do we need an order of operations? Here is the beginning of a famous piece of music:



What happens to that music if we don't know the order in which to read or write it? What if Beethoven had not known the correct notation?

Any language has its own conventions. If you saw the sentence: "The cat mat on the sat", you would notice that as an English expression it is really bad (or that there is a typo)! The same goes for a mathematical expression. If we do not follow conventions, accepted world-wide, we will not be able to communicate effectively.

As long as we understand the ORDER of the operations, we can express and understand any mathematics we like, no matter how complicated.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION

Let's start with some "straightforward" conventions.

Imagine you are helping a young child keep tabs on their money. A generous grandparent gave them \$10 for their birthday. You supplement it with a little pocket money when you want to. This might be a record of their "transactions":

$$10 - 4 + 2 - 7 + 2$$

The result of these transactions is that they now have the grand total of \$3. This is a conventionally written mathematical expression; we do each calculation from left to right as it occurs. So the child starts with \$10, spends \$4 (10 subtract 4 leaves 6), receives \$2 (6 add 2 gives 8), spends \$7 (8 subtract 7 gives 1) then receives \$2 (1 add 2 gives the final result of 3). Actually, just about all calculators can handle this transaction: simply type in the expression as it is written.

The only way we can change the order of operations when there is only $+$ and/or $-$ is to insert $(\)$. More about this later. Otherwise, just calculate from left to right.

Now, imagine you are at the checkout at the supermarket. The bill has come to \$27. Just before you pay, you see a treat that you think you'll buy for the children. Each treat costs \$2 and you have 3 children. The invoice will now read (without the \$ signs):

$$27 + 3 \times 2$$



In other words, keep the 27 on hold, calculate the 3×2 , then add, to get $27 + 6 = 33$. So your bill is actually \$33. This is the next convention for the order of operations. Multiplication and division are both done before addition and subtraction, but still in order from left to right. So:

$$5 + 6 \times 3 - 4$$

is done as:

$$\begin{aligned} &5 + 18 - 4 \\ &= 23 - 4 \\ &= 19. \end{aligned}$$

(Using the supermarket example: your running total is \$5, you buy 6 lots of an item costing \$3, and you decide you don't want something that cost \$4.)

Again at the supermarket, you buy an item which is "half-price". Your running total is \$19, the half-price item originally cost \$10. The notation would be:

$$19 + 10 \div 2$$

The calculation is done by keeping the 19 on hold, calculating $10 \div 2$ then adding, so:

$$19 + 5 = 24.$$

THE STORY SO FAR:

- If you only have + and −, work from left to right. (Neither operation takes precedence over the other.)
- If you only have \times and \div , work from left to right. (Neither operation takes precedence over the other.)
- If you have a mixture of +, −, \times and \div , do the \times and \div first in order from left to right, followed by the + and − in order from left to right.

So:

$$15 - 6 \div 2 + 8 \times 3 - 10 \div 5$$

is done by holding onto the 15, calculating $6 \div 2$ (which gives 3), calculating 8×3 (which gives 24), and calculating $10 \div 5$ (which gives 2), so we get:

$$15 - 3 + 24 - 2,$$

which we do from left to right to get 34.

Any scientific calculator should do this calculation in the conventional (correct) order; you just type in what is written and the calculator sorts out the order.

INTRODUCING BRACKETS

Let us go back to our child's transactions.

This time, let's insert some brackets (in maths these round brackets are actually called "parentheses", but let's stick to "brackets" here).

$$10 - (4 + 2) - 7 + 2$$

This means that the child actually spent a total of \$6 (from $4 + 2$) in their first spending spree! The calculation changes to $10 - 6 - 7 + 2$. Notice they have gone into debt! Left to right: $10 - 6 = 4$, and $4 - 7 = -3$. Even your \$2 pocket money doesn't help them get out of debt: $-3 + 2 = -1$.

Try this one:

$$10 - (4 + 2) - (7 + 2).$$

You should get −5, because you would do:

$$10 - 6 - 9 = 4 - 9 = -5.$$



Notice how the brackets really change things. In fact, brackets take precedence over all the other operations. Always do what is in the brackets first.

POWERS AND ROOTS

One further operation is “raising a number to a power”, such as “squaring” or “cubing”. When we raise a number to a power, we are really just multiplying that number by itself a given number of times.

So, for example, 3^2 (read as “three squared”) means 3×3 , (which is 9), 2^4 (read as “two to the power four” or “two to the fourth”) means $2 \times 2 \times 2 \times 2$ (which is 16), $3^2 \times 2^4$ means 9×16 (which is 144).

Raising a number to a power takes precedence over all other operations except brackets.

So:

- $8^2 + 4^2 = 8 \times 8 + 4 \times 4 = 64 + 16 = 80$
- $8^2 \times 4^2 = 64 \times 16 = 1024$
- $8^2 - 4^2 = 64 - 16 = 48$
- $8^2 \div 4^2 = 64 \div 16 = 4$

But

$$(8 + 4)^2 = 12^2 = 144$$

We are just about there! Now let’s look at square roots, $\sqrt{\quad}$. Taking a square root is the same as raising to the power of a half, but everything under the square root symbol must be raised to the power (use brackets), that is $\sqrt{a + b} = (a + b)^{\frac{1}{2}}$. So:

- $\sqrt{16} = 4$ because $\sqrt{16}$ asks us to look for the number that multiplies by itself to give 16; we do the inverse of squaring. 4×4 is 16, so $\sqrt{16}$ is 4.
- $\sqrt{4 + 5} = \sqrt{9}$ because the root sign is over the complete addition calculation and acts like a pair of brackets. Also, $\sqrt{9} = 3$ because we find the number which multiplies by itself to give 9. So, $\sqrt{4 + 5} = 3$.
- However, $\sqrt{4} + 5 = 2 + 5 = 7$, because we only find the square root of 4 this time (2), before adding it to the 5.

RECAP

- **Brackets first, then**
- **Raising to powers and finding roots, in order from left to right, then**
- **Multiplication and division next, in order from left to right, and last,**
- **Addition and subtraction, in order from left to right.**

Combining:

$$\begin{aligned}(5 - 3 + 1)^2 \times (26 - 2) \div 8 - (7 - 3) + \sqrt{3^2 + 4^2} \\ &= 3^2 \times 24 \div 8 - 4 + \sqrt{9 + 16} \\ &= 9 \times 24 \div 8 - 4 + \sqrt{25} \\ &= 216 \div 8 - 4 + 5 \\ &= 27 - 4 + 5 \\ &= 28\end{aligned}$$



One more thing: If you see something like $\frac{6+2}{4}$, the line separating the top expression from the bottom acts like a pair of brackets, so you do the $6 + 2$ first (8) then divide it by 4 (result is 2).

Here are some for you to try: (Fully worked solutions are attached.) You can also check these using a scientific calculator, simply typing in what you see; the calculator will worry about the correct order.

EXERCISES

1. $8 + 3 - 4 + 2$

2. $9 \div 3 + 2 \times 6$

3. $10 \div (2 + 3)$

4. $4^2 + 3^2 - 2^2$

5. $(4 + 3)^2$

6. $\frac{6+4}{5-3}$

7. $6 \times (8 - 3) - 3 \times (2 + 3)$

8. $24 \div (19 - 3 \times 5)$

9. $(9 - (6 - 3)) \times (8 + 4)$

10. $2 + \sqrt{7 - 6 + 8} \times (14 - 2 \times 5)^2$

If you need help with any of the maths covered in this resource (or any other maths topics), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.



SOLUTIONS TO EXERCISES

1. Work from left to right:

$$\begin{aligned}8 + 3 - 4 + 2 &= 11 - 4 + 2 \\ &= 7 + 2 \\ &= 9\end{aligned}$$

2. Multiplication and division first, from left to right, then add

$$\begin{aligned}9 \div 3 + 2 \times 6 &= 3 + 12 \\ &= 15\end{aligned}$$

3. Brackets first

$$\begin{aligned}10 \div (2 + 3) &= 10 \div 5 \\ &= 2\end{aligned}$$

4. Powers first then add and subtract from left to right

$$\begin{aligned}4^2 + 3^2 - 2^2 &= 16 + 9 - 4 \\ &= 25 - 4 \\ &= 21\end{aligned}$$

5. Brackets first

$$\begin{aligned}(4 + 3)^2 &= 7^2 \\ &= 49\end{aligned}$$

6. Calculate top (numerator) and bottom (denominator) first

$$\begin{aligned}\frac{6 + 4}{5 - 3} &= \frac{10}{2} \\ &= 5\end{aligned}$$

7. Brackets first, then multiplication

$$\begin{aligned}6 \times (8 - 3) - 3 \times (2 + 3) &= 6 \times 5 - 3 \times 5 \\ &= 30 - 15 \\ &= 15\end{aligned}$$

8. Brackets first, with multiplication first in the brackets!

$$\begin{aligned}24 \div (19 - 3 \times 5) &= 24 \div (19 - 15) \\ &= 24 \div 4 \\ &= 6\end{aligned}$$

9. Don't be scared by all the brackets. Do them first, in their pairs, from the inside out.

$$\begin{aligned}(19 - (6 - 3)) \times (8 + 4) &= (19 - 3) \times 12 \\ &= 16 \times 12 \\ &= 192\end{aligned}$$



10. A review of all the rules.

$$\begin{aligned}2 + \sqrt{7 - 6 + 8} \times (14 - 2 \times 5)^2 &= 2 + \sqrt{1 + 8} \times (14 - 10)^2 \\ &= 2 + \sqrt{9} \times 4^2 \\ &= 2 + 3 \times 16 \\ &= 2 + 48 \\ &= 50\end{aligned}$$

