Fractions

2. Multiplication and Division

If you have not worked with fractions for a while, you might like to begin by looking at *Fractions – 1. Manipulating Fractions*.

MULTIPLICATION

THE ALGORITHM

Multiplication is probably the simplest operation to do with fractions as the algorithm is the simplest. To multiply two fractions, you just multiply the two numerators together and multiply the two denominators together. Sometimes the resulting fraction can then be simplified by finding an equivalent fraction (if you are not sure how to simplify fractions, please have a look at *Fractions – 1. Manipulating Fractions*).

Let's go straight to some examples:

$$\begin{aligned}\frac{1}{2}\times\frac{1}{3} &= \frac{1\times 1}{2\times 3}\\ &= \frac{1}{6}\end{aligned}$$

In the next example we will have to simplify our fraction after multiplication, this shows that even if you start with simplified fractions, your result may not be.

$$\begin{aligned}\frac{2}{5}\times\frac{3}{8} &= \frac{2\times 3}{5\times 8}\\ &= \frac{6}{40}\\ &= \frac{6\div 2}{40\div 2}\\ &= \frac{3}{20}\end{aligned}$$

In our last example we consider multiplying a fraction by a whole number, we do this by turning the whole number into a fraction.

$$\begin{aligned}4\times\frac{2}{3} &= \frac{4}{1}\times\frac{2}{3}\\ &= \frac{4\times 2}{1\times 3}\\ &= \frac{8}{3}\end{aligned}$$

In this last case we also obtained an improper fraction, we could also write this as a mixed number and get $2\,\frac{2}{3}$.

THE MODEL

Now let's look at why this algorithm works. When we read multiplication we often use the descriptor "of" or "lots of". For example 3×2 is "three lots of two". With fractions it is the same: $\frac{1}{2}\times\frac{1}{3}$ is "one half of one third". Let's see what that looks like, here is one third:

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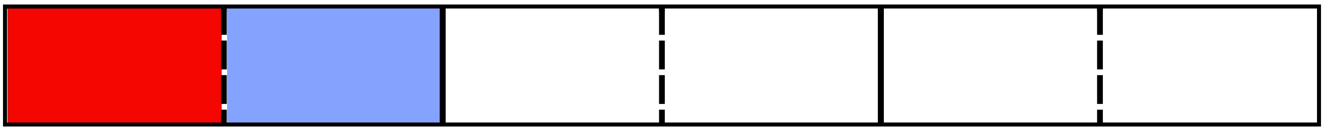
And we want one half of this. We can represent this two ways, the first is by breaking up our whole horizontally:

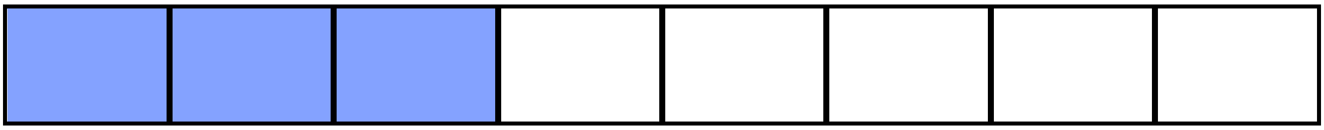
The area in the intersection of our horizontal and vertical shading is then the result of "one half of one third". We see that our whole is now broken into 6 pieces and our double shaded area is just one piece. Therefore we are left with $\frac{1}{6}$ as our answer to $\frac{1}{2} \times \frac{1}{3}$.

The second way we can take one half of our one third is by breaking up our shaded area into 2 pieces and taking 1 of these:

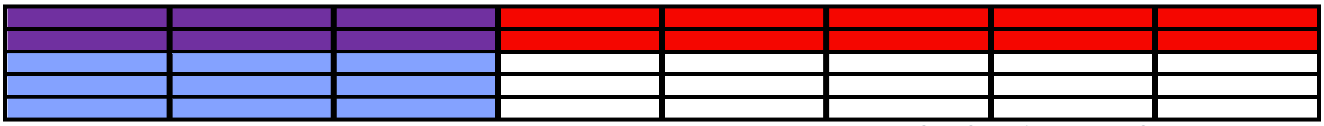


Note that we also had to break each of the other (non-shaded) pieces into 2 to find out our denominator (but we don't shade any of them).

Let's consider the next example: $\frac{2}{5} \times \frac{3}{8}$ and start by drawing our $\frac{3}{8}$.



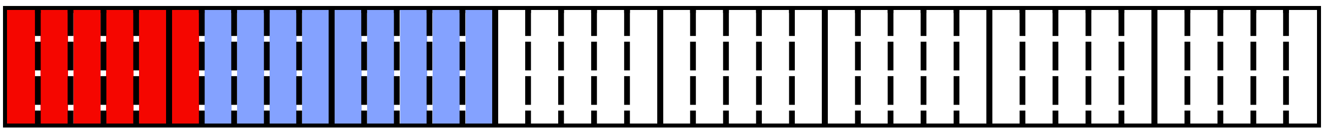
Now we divide the whole horizontally to represent $\frac{2}{5}$.



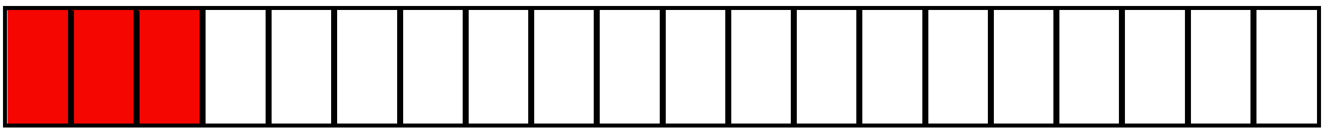
We can see that 6 pieces out of the 40 pieces there are shaded both blue and red, so $\frac{2}{5} \times \frac{3}{8} = \frac{6}{40}$ (that is, $\frac{3}{20}$, after dividing the numerator and denominator by 2).

This calculation can also be done the second way. In this case we break each piece up into 5 parts and take 2 of those smaller pieces if the original piece was shaded:

We again see that there are 40 pieces in total and 6 have been shaded red. Of course we would still need to simplify this fraction, and with this method we could also try and do this visually. We first rearrange so that the colours are together:

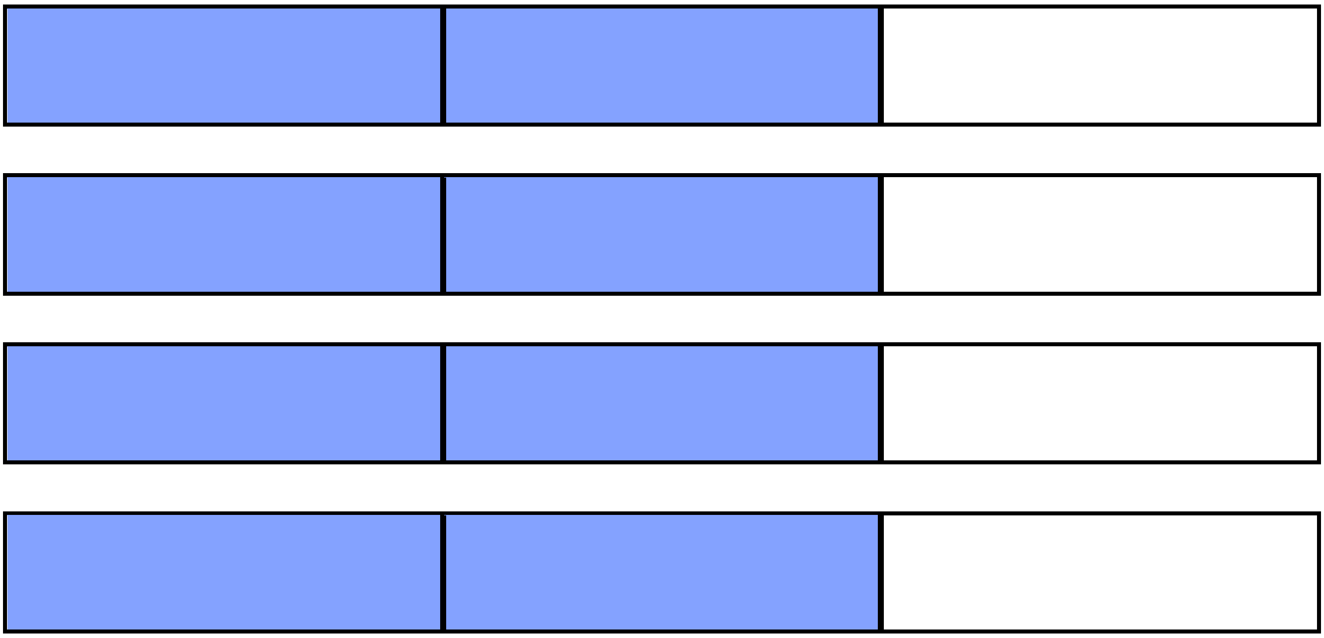


Next we remove the original blue shading and see if we can remove some of the break points:

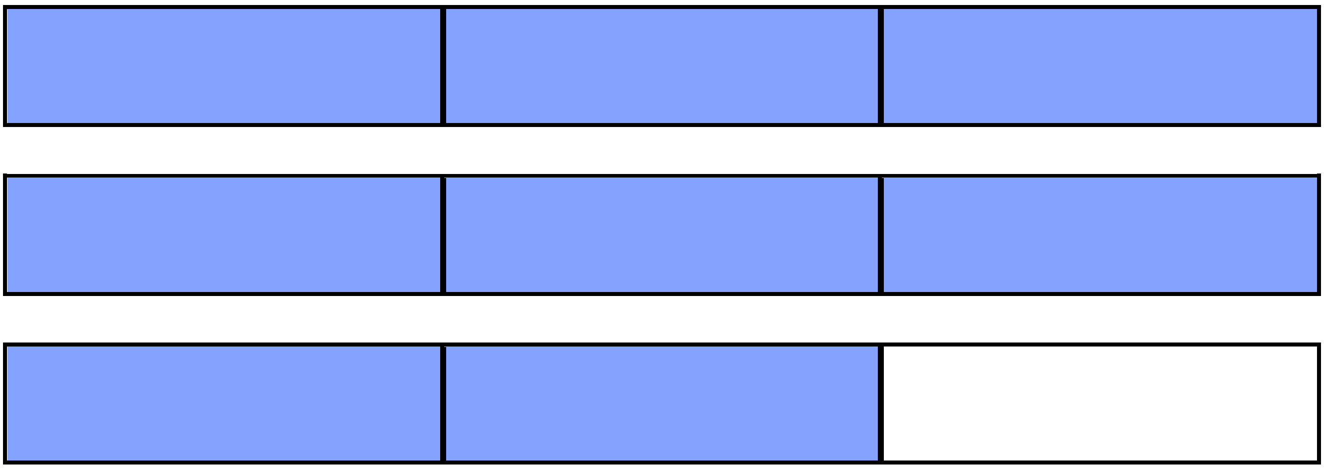


We see here that we are left with $\frac{3}{20}$.

For the last example, $4 \times \frac{2}{3}$ we can use the same "lots of" method that we used when only dealing with whole numbers. That is we draw 4 lots of $\frac{2}{3}$.



We can see here that all up we have 8 shaded regions, each of which are a third of the whole, hence the answer is $\frac{8}{3}$. We can also rearrange the pieces we have, for example by putting the last two shaded regions into the empty slots of the first two diagrams we obtain:



So altogether we have two complete wholes, and an extra two thirds. So our answer as a mixed number is $2\,\frac{2}{3}$.

CROSS CANCELLATION

Here we look at a shortcut in performing multiplication of fractions, but beware this technique only works when you are about to multiply.

In *Fractions – 1. Manipulating Fractions* we saw how we can simplify fractions by dividing the top and bottom of a fraction by the same number. When multiplying fractions we can do a special simplification before multiplying called cross cancellation, in this case we divide the top of one fraction and the bottom of the other by the same number. It doesn't matter which fraction the top comes from and which the bottom comes from (just as long you don't do two tops or two bottoms). For example:

$$\frac{3}{4} \times \frac{2}{7} = \frac{3}{4 \div 2} \times \frac{2 \div 2}{7}$$
$$= \frac{3}{2} \times \frac{1}{7}$$
$$= \frac{3}{14}$$

We noticed that 2 divides evenly into the denominator of the first fraction and the numerator of the second, so this allowed us to simplify before doing the multiplication. This is helpful in reducing large multiplications, for example:

$$\frac{49}{81} \times \frac{36}{77}$$

In this case doing the multiplications 49×36 and 81×77 can take some time (unless you have a calculator), and after doing this multiplication we would then have to simplify our fraction. Instead we notice that we can cross cancel by dividing the numerator of the first fraction and denominator of the second fraction by 7, and the denominator of the first fraction and the numerator of the second fraction by 9;

$$\frac{49}{81} \times \frac{36}{77} = \frac{49 \div 7}{81 \div 9} \times \frac{36 \div 9}{77 \div 7}$$
$$= \frac{7}{9} \times \frac{4}{11}$$
$$= \frac{28}{99}$$

So the procedure in multiplying fractions is to simplify each fraction, perform any cross cancellations, multiply tops and bottoms, before finally simplifying the resulting fraction (if you fully simplify and cross cancel beforehand you will not need to simplify afterwards).

Here are some for you to try. You can check your answers with the solutions at the end.

EXERCISES

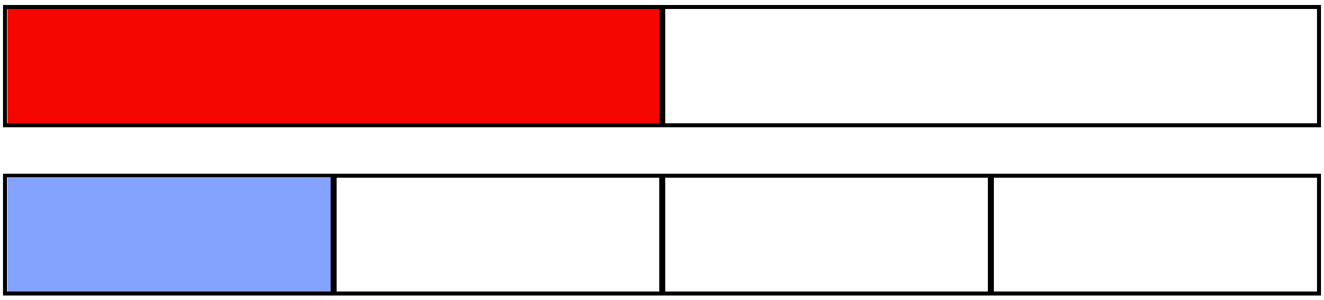
1. $\frac{3}{4} \times \frac{2}{3}$
2. $\frac{1}{2} \times \frac{3}{5}$
3. $\frac{35}{24} \times \frac{9}{28}$
4. $5 \times \frac{1}{2}$

DIVISION

THE MODEL

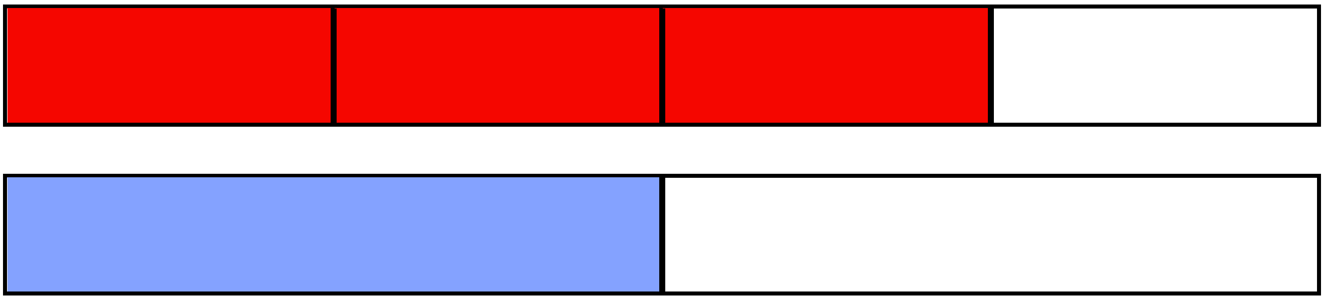
While multiplication is the easiest of the fraction operations (at least in terms of the algorithm), division is not far behind. Let's first consider using our visualisation model. When dividing two numbers we ask ourselves "how many?" For example $12 \div 3$ is the same as asking "how many threes in twelve?" And of course the answer is 4. It is no different with fractions.

For example, $\frac{1}{2} \div \frac{1}{4}$ would read as "how many quarters in one half?" If we look at our fraction visualisations:



We can see here that we would need two of the quarter pieces to fill the space of a half, so the answer is 2.

Let's look at another example. $\frac{3}{4} \div \frac{1}{2}$ means "how many halves in three quarters?" Visualising the two fractions we have:



We can see from this that there are one and a half $\frac{1}{2}$s in $\frac{3}{4}$ so our answer is $1\,\frac{1}{2}$.

THE ALGORITHM

How can we do this division for general numbers though? We remember here that division is the inverse of multiplication, so we can use our multiplication knowledge to divide. For example, when we calculated $12 \div 3$ we asked "how many threes in twelve?" We then remembered our multiplication tables and realised that $4 \times 3 = 12$, so that there are 4 lots of 3 in 12, that is $12 \div 3 = 4$.

Now suppose instead of dividing by 3 we try and work out $12 \times \frac{1}{3}$? We would write this as $\frac{12}{1} \times \frac{1}{3}$, and that gives $\frac{12}{3}$, which simplifies to 4. The same answer we got from our division! In fact, recall that another way of interpreting a fraction is as division, so we could read $\frac{12}{3}$ as twelve divided by three. Hence $12 \div 3 = 12 \times \frac{1}{3}$.

This technique of changing division to multiplication can be used in all our division problems. Our method is to flip the second fraction and change the division sign to a multiplication sign.

Let's reconsider the first example above: $\frac{1}{2} \div \frac{1}{4}$. We flip the $\frac{1}{4}$ and change the division to multiplication to obtain $\frac{1}{2} \times \frac{4}{1}$. We can now perform the multiplication just as before:

$$\begin{aligned} \frac{1}{2} \times \frac{4}{1} &= \frac{4}{2} \\ &= 2. \end{aligned}$$

Hence our answer is 2, which is what we also got when using our visual models. Let's try the second example:

$$\begin{aligned} \frac{3}{4} \div \frac{1}{2} &= \frac{3}{4} \times \frac{2}{1} \\ &= \frac{3 \times 2}{4 \times 1} \\ &= \frac{6}{4} \\ &= \frac{6 \div 2}{4 \div 2} \\ &= \frac{3}{2}. \end{aligned}$$

And $\frac{3}{2}$ is the same as $1\,\frac{1}{2}$.

You might remember learning in school to "*invert and multiply*". This is exactly how we do division. We leave the first number alone, change the symbol from ÷ (division) to × (multiplication), and flip (invert) the second number. We can then perform the multiplication using all the tricks we learnt in the previous section (make sure you only apply any cross cancellations after you have flipped the second fraction).

One final example, $5 \div \frac{1}{10}$. We could read this as "how many tenths in five?" And since there are 10 tenths in one, there must be 50 tenths in five, but let's use our method.

$$
\begin{aligned}
5 \div \frac{1}{10} &= 5 \times \frac{10}{1} \\
&= \frac{5}{1} \times \frac{10}{1} \\
&= \frac{50}{1} \\
&= 50.
\end{aligned}
$$

Here are some for you to try. You can check them with the worked solutions at the end.

EXERCISES

5. $\frac{1}{3} \div \frac{1}{2}$

6. $\frac{3}{5} \div \frac{3}{10}$

7. $9 \div \frac{1}{3}$

8. $\frac{3}{4} \div \frac{1}{8}$

If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

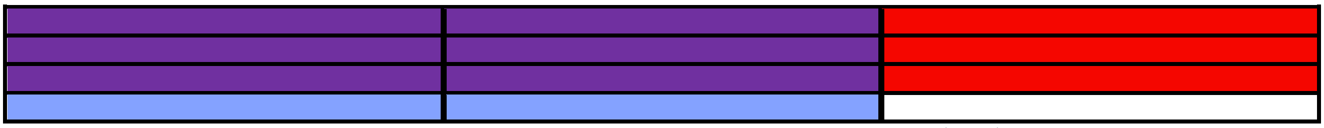
1. $\frac{3}{4} \times \frac{2}{3}$

$$\begin{aligned}\frac{3}{4} \times \frac{2}{3} &= \frac{3 \times 2}{4 \times 3} \\ &= \frac{6}{12} \\ &= \frac{6 \div 6}{12 \div 6} \\ &= \frac{1}{2}.\end{aligned}$$

Note we could have also cross cancelled before doing the multiplication (but as the numbers were small it wasn't necessary).

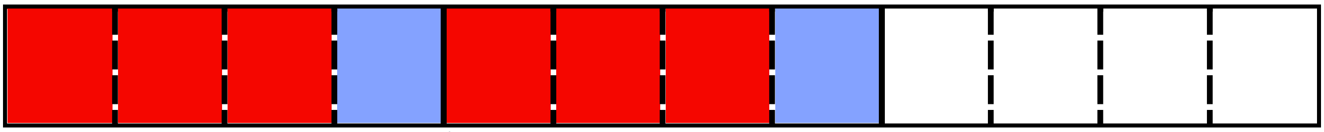
$$\begin{aligned}\frac{3}{4} \times \frac{2}{3} &= \frac{3 \div 3}{4} \times \frac{2}{3 \div 3} \\ &= \frac{1}{4 \div 2} \times \frac{2 \div 2}{1} \\ &= \frac{1}{2} \times \frac{1}{1} \\ &= \frac{1}{2}.\end{aligned}$$

You may have also completed the problem using visualisations. For example, the first method gives:

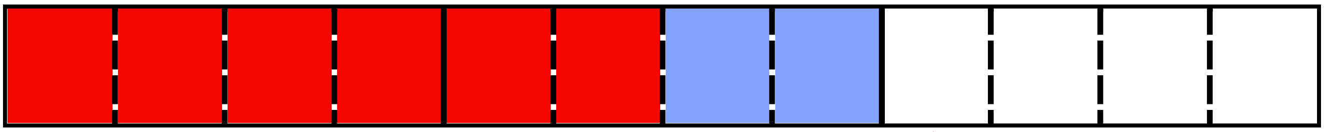


To see that 6 of the 12 pieces have been shaded twice, which is the same as $\frac{6}{12} = \frac{1}{2}$.

The second visualisation method gives:



Which we again recognise as $\frac{6}{12}$, or collecting the colours we obtain:

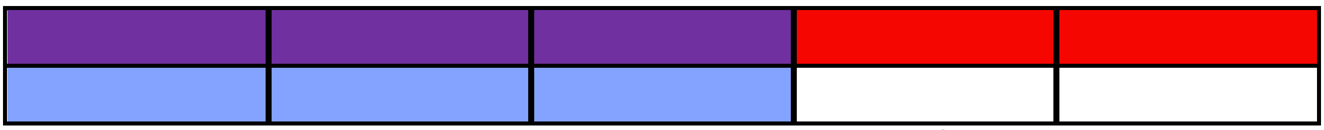


From which we see the red part takes up half the total area so our answer is $\frac{1}{2}$.

2. $\frac{1}{2} \times \frac{3}{5}$

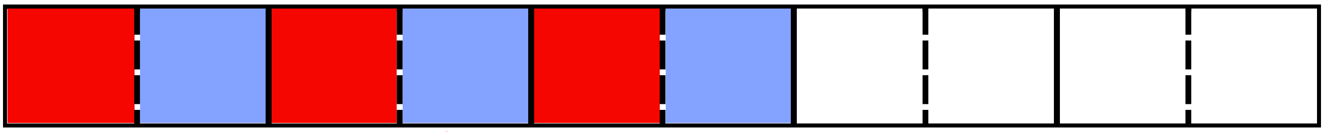
$$\begin{aligned}\frac{1}{2} \times \frac{3}{5} &= \frac{1 \times 3}{2 \times 5} \\ &= \frac{3}{10}\end{aligned}$$

You may have also completed the problem using visualisations. For example, the first method gives:



To see that 3 of the 10 pieces have been shaded twice, which is the same as $\frac{3}{10}$.

The second visualisation method gives:



Which we again recognise as $\frac{3}{10}$.

3. $\frac{35}{24} \times \frac{9}{28}$

In this question the numbers are quite large so we first look to see if we can simplify either fraction. We can't, so we move on to cross cancellation. We note that both 24 and 9 are divisible by 3, and that both 35 and 28 are divisible by 7.

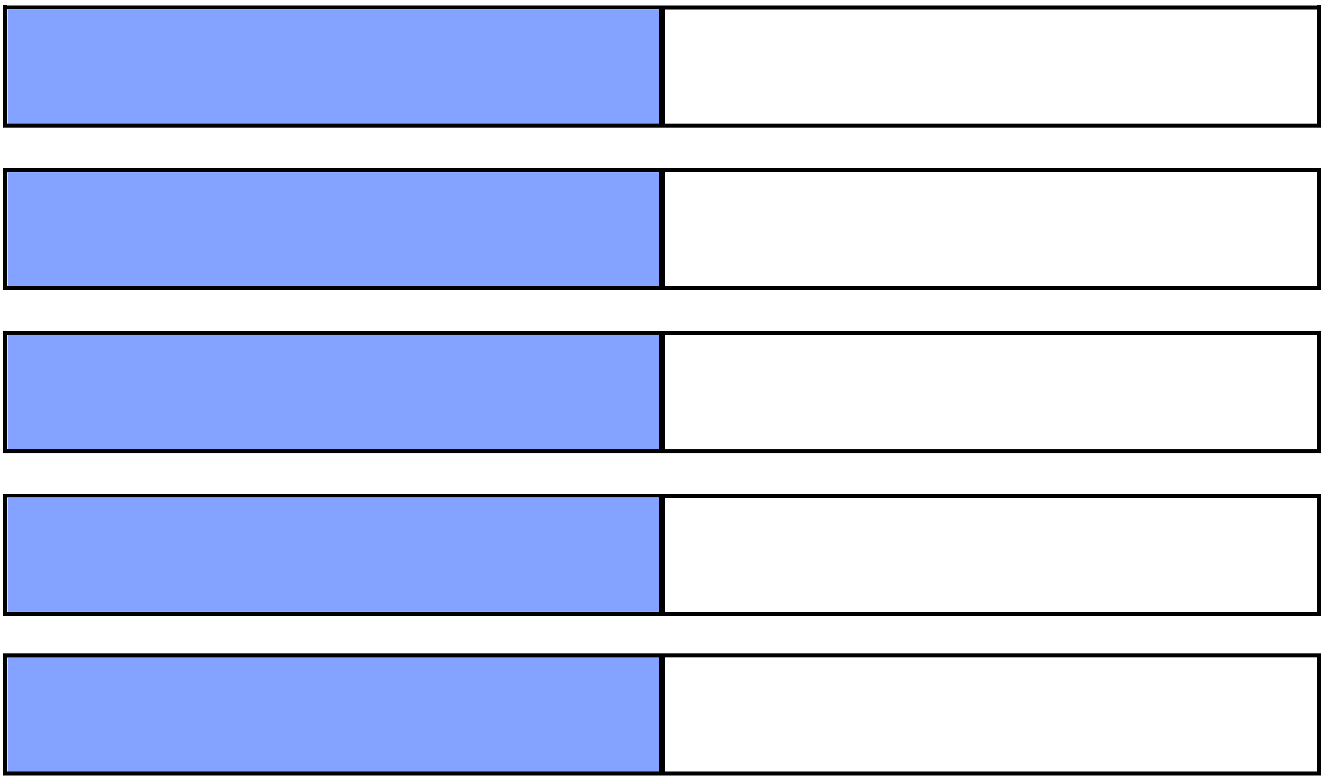
$$\begin{aligned} \frac{35}{24} \times \frac{9}{28} &= \frac{35}{24 \div 3} \times \frac{9 \div 3}{28} \\ &= \frac{35 \div 7}{8} \times \frac{3}{28 \div 7} \\ &= \frac{5}{8} \times \frac{3}{4} \\ &= \frac{15}{32}. \end{aligned}$$

We won't do the visual methods from this problem as there would be so many divisions (although you could do this after you have cross cancelled and obtained the $\frac{5}{8} \times \frac{3}{4}$.

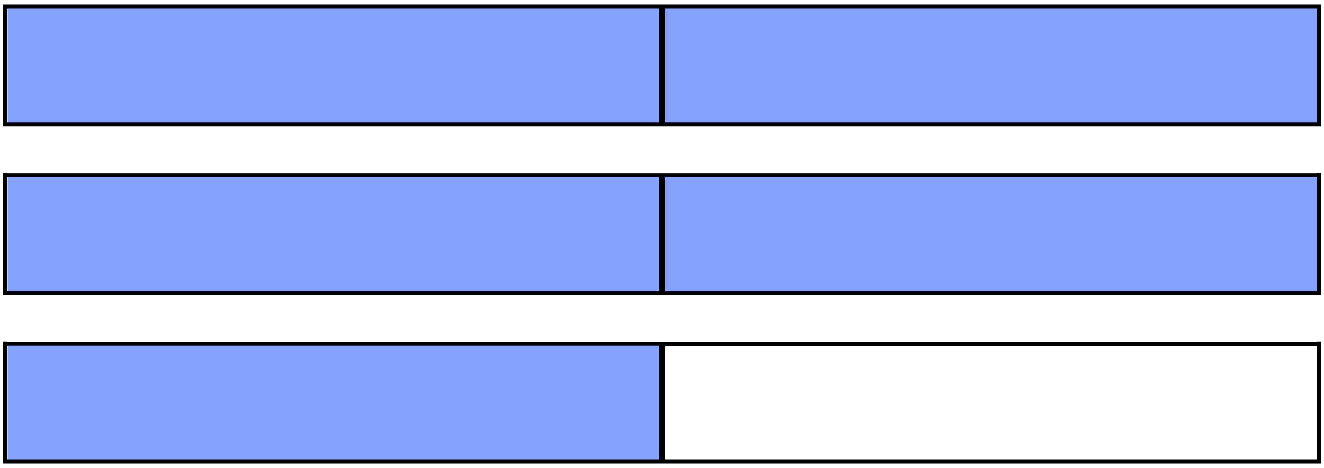
4. $5 \times \frac{1}{2}$

$$\begin{aligned} 5 \times \frac{1}{2} &= \frac{5}{1} \times \frac{1}{2} \\ &= \frac{5 \times 1}{1 \times 2} \\ &= \frac{5}{2} \end{aligned}$$

Which we could also write as a mixed number: $2\frac{1}{2}$. Using the visualisation model we draw 5 lots of $\frac{1}{2}$:



And by collecting all the shaded parts we obtain:



Which we can see is $\frac{5}{2} = 2\,\frac{1}{2}$.

5. $\frac{1}{3} \div \frac{1}{2}$

$$\begin{aligned}\frac{1}{3} \div \frac{1}{2} &= \frac{1}{3} \times \frac{2}{1} \\ &= \frac{1 \times 2}{3 \times 1} \\ &= \frac{2}{3}.\end{aligned}$$

6. $\frac{3}{5} \div \frac{3}{10}$

$$\begin{aligned}\frac{3}{5} \div \frac{3}{10} &= \frac{3}{5} \times \frac{10}{3} \\ &= \frac{3 \times 10}{5 \times 3} \\ &= \frac{30}{15} \\ &= 2.\end{aligned}$$

Of course we could have done the multiplication step using cross cancellation by dividing the first numerator and second denominator by 3 and the first denominator and second numerator by 5 to get:

$$\begin{aligned}\frac{3}{5} \div \frac{3}{10} &= \frac{3}{5} \times \frac{10}{3} \\ &= \frac{3 \div 3}{5} \times \frac{10}{3 \div 3} \\ &= \frac{1}{5 \div 5} \times \frac{10 \div 5}{1} \\ &= \frac{1}{1} \times \frac{2}{1} \\ &= \frac{2}{1} \\ &= 2.\end{aligned}$$

But this way is actually longer!

7. $9 \div \frac{1}{3}$

$$\begin{aligned}9 \div \frac{1}{3} &= \frac{9}{1} \times \frac{3}{1} \\ &= \frac{27}{1} \\ &= 27.\end{aligned}$$

8. $\frac{3}{4} \div \frac{1}{8}$

$$\begin{aligned}\frac{3}{4} \div \frac{1}{8} &= \frac{3}{4} \times \frac{8}{1}\\ &= \frac{3 \times 8}{4 \times 1}\\ &= \frac{24}{4}\\ &= 6.\end{aligned}$$

Again we could have used cross cancellation immediately before the multiplication step, remember to be careful not to use it before you flip the second fraction though.