Fractions

1. Manipulating Fractions

INTRODUCTION

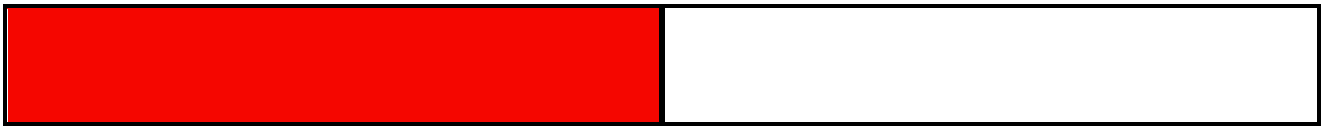
Firstly, let's think about what a fraction is. A fraction consists of a *numerator* (the top) and a *denominator* (the bottom):

$$\frac{\boldsymbol{numerator}}{\boldsymbol{denominator}}$$

One way to look at what this means is the "part of a whole" definition:

- The *denominator* represents the number of pieces the whole has been divided into
- The *numerator* represents the number of pieces we have/are interested in

We can also represent this information visually, for example:



Looking at the shaded section, we could represent this as the fraction $\frac{1}{2}$. The denominator is 2 because the whole strip has been divided into 2 pieces. The numerator is 1 because we are interested in one of those pieces (the shaded piece).

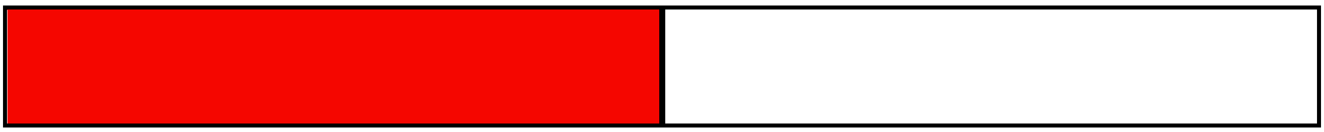
Another way to think about fractions is as division. The line between the numerator and denominator is another way of writing the symbol ÷. So we can think of a fraction as the top number (numerator) being divided by the bottom number (denominator), or "how many times does the denominator go into the numerator?" So, for example, $\frac{1}{2}$ is the same as $1 \div 2$.

Note that we can also write fractions with the separation line at an angle. For example we can write one half as $^1/_2$ instead of $\frac{1}{2}$.

PROPER, IMPROPER, AND MIXED

PROPER FRACTIONS

Think about our first definition of a fraction. We have a number of pieces out of all the pieces that a whole has been divided into. So, if the fraction we are looking at has the numerator <u>smaller</u> than the denominator, for example, $\frac{1}{2}$ or $\frac{2}{3}$, then we have fewer pieces than the whole is made up of. Such a fraction is called *proper*.



(1 piece is shaded out of 2 so the fraction is $\frac{1}{2}$)

(2 pieces are shaded out of 3 so the fraction is $\frac{2}{3}$)

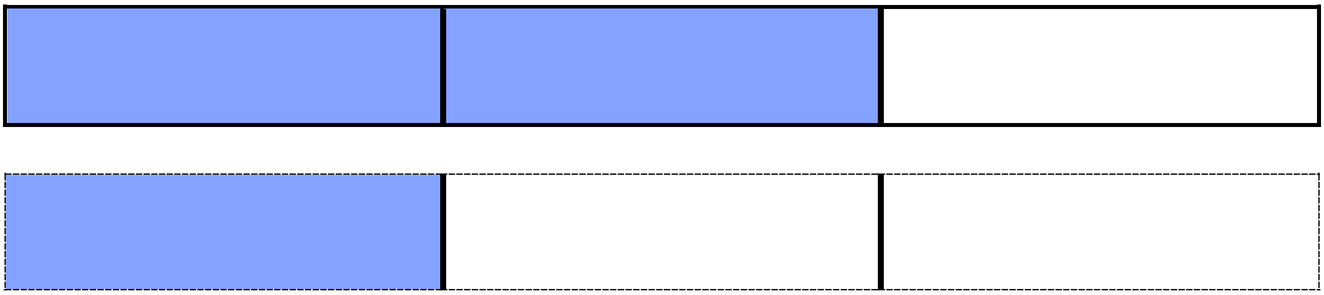
*A fraction where the **numerator is smaller than the denominator** is called a **proper fraction**.*

If we use our second way of thinking of a fraction, then a fraction is proper if when we try and divide the numerator by the denominator, we only get 0 and some remainder. For example, with $\frac{2}{3}$, we would say “how many times will 3 go into 2?” It will not go at all and so $\frac{2}{3}$ is a proper fraction.

IMPROPER FRACTIONS

If the numerator is <u>bigger</u> than the denominator, for example $\frac{4}{3}$, then the denominator will divide into the numerator at least once. This type of fraction is called an *improper* fraction.

*A fraction where the **numerator is larger than the denominator** is called an **improper fraction**.*



$\frac{4}{3}$ is the same as $1\frac{1}{3}$ (read as “one and one third”) because 3 divides into 4 once (1 whole) and there is 1 piece left over (which is the one third piece). You can see this from the diagram.

We call $1\frac{1}{3}$ a **mixed number**, because it is made up of a whole number and a fraction. We can only make mixed numbers from improper fractions, not from proper fractions.

WHOLE NUMBERS (INTEGERS)

Just quickly let us reconsider whole numbers, i.e. 1, 2, 3, 4, 5, etc. These are numbers where we haven’t split anything into parts (they are still whole). Another way to say this is that there is just one “part”. This means that, if we want to, we can write a whole number as a fraction by putting a 1 in the denominator.

So, 1 is the same as $\frac{1}{1}$, 2 is the same as $\frac{2}{1}$, 3 is the same as $\frac{3}{1}$, 137 is the same as $\frac{137}{1}$, and so on, even 0 is the same as $\frac{0}{1}$. This is helpful when we want to apply operations (e.g. when we are adding, subtracting, multiplying and dividing) and fractions are involved.

Here are some exercises for you to try. You can check your answer with the solutions at the end of the resource.

EXERCISES

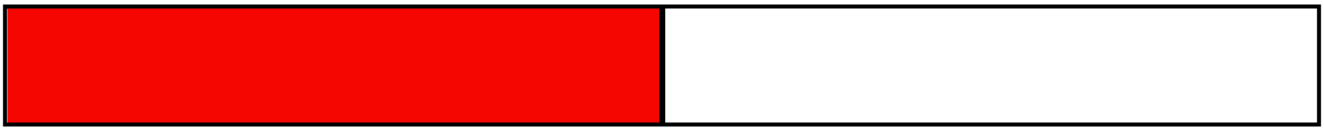
Are these fractions proper or improper? If the fraction is improper, write the equivalent mixed number.

1. $\frac{5}{8}$
2. $\frac{5}{4}$
3. $\frac{3}{10}$
4. $\frac{17}{6}$

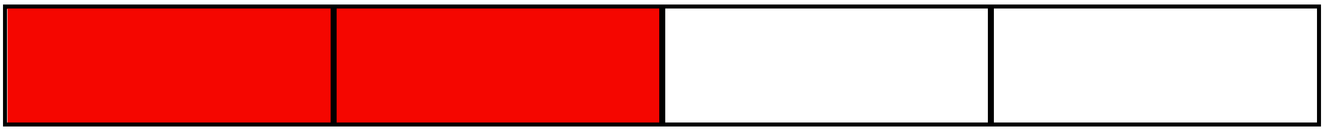
EQUIVALENT FRACTIONS

As seen in the case of whole numbers there are many ways to write the same number. This is true even when just considering writing a number as a fraction. Can you name any fractions that are the same as (equivalent to) $\frac{1}{2}$?

Let's reconsider what one half looks like visually:



The important thing here is the actual shaded area, if we split the whole into more pieces without changing the amount of shaded area, then we obtain an equivalent fraction. For example:



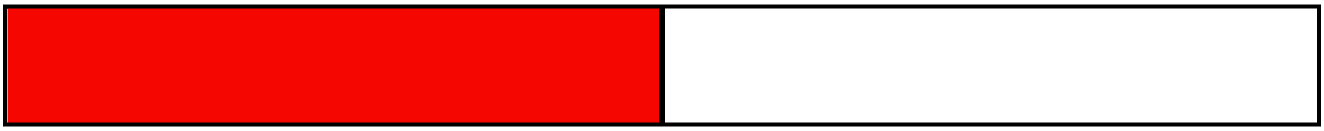
These diagrams tell us that $\frac{1}{2}$ is the same as $\frac{2}{4}$!

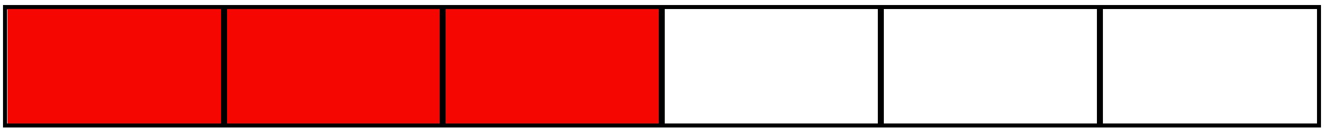
How else can we obtain this equivalence? Well we can also obtain $\frac{2}{4}$ from $\frac{1}{2}$ by multiplying both the numerator and the denominator by 2. So,

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

That is, if we double the number of parts in our whole, but double the number of pieces we have, nothing changes.

We can also obtain $\frac{3}{6}$ as an equivalent fraction for $\frac{1}{2}$. Using visuals we can see this:





And using our algorithm of multiplying top and bottom by the same number, we can also see this:

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$

In fact, this shows us there is an infinite set of fractions that are equivalent to $\frac{1}{2}$, or indeed any fraction! We can obtain them by multiplying the top and bottom by the same whole number. For example, $\frac{1}{2}$ is equivalent to $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots, \frac{25}{50}, \ldots$ etc. While $\frac{2}{3}$ is equivalent to $\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots, \frac{50}{75}, \ldots$ etc.

We can also work backwards. That is reduce the number of pieces our whole is divided into (be careful not to change the shaded area), which is the same as dividing the numerator and denominator by the same number. For example, we could start at $\frac{3}{6}$, and divide both the numerator and denominator by 3 (notice that 3 is a factor of each number, i.e. it divides each number with zero remainder) to get $\frac{3 \div 3}{6 \div 3} = \frac{1}{2}$. Let's have a look at this using our fraction visualisation:

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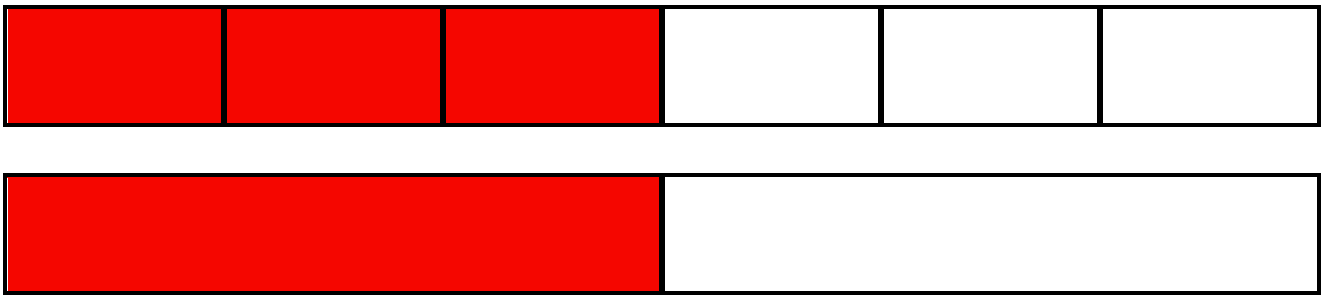


Dividing the numerator and denominator by the same number is the way to write a fraction in its *simplest form*, that is, you obtain the lowest possible values for the numerator and denominator for that particular number. This is called *simplifying* and fraction. You can check whether a number is in its simplest form by checking if the only number that divides both the numerator and denominator is 1.

CALCULATING EQUIVALENT FRACTIONS

FINDING MISSING NUMBERS

Sometimes you will need to work out an equivalent fraction that has a specific numerator or denominator. For example, you might have the fraction $\frac{2}{3}$ and need to find the equivalent fraction that has a denominator of 12, that is you need to fill in the box in the following equation:

$$\frac{2}{3} = \frac{\square}{12}$$

To do this you compare the denominators (or numerators in the case where you know the new numerator) of both fractions. 3 has been multiplied by 4 to make 12, so to balance the fraction we also need to multiply the numerator, 2, by 4:

$$\begin{aligned} \frac{2}{3} &= \frac{2 \times 4}{3 \times 4} \\ &= \frac{8}{12}. \end{aligned}$$

Here is a case where you are given a new numerator.

$$\frac{3}{2} = \frac{9}{\square}$$

In this case we compare numerators and see that the 3 has been multiplied by 3 to give 9, thus we need to multiply to denominator by 3 to obtain the equivalent fraction:

$$\begin{aligned} \frac{3}{2} &= \frac{3 \times 3}{2 \times 3} \\ &= \frac{9}{6}. \end{aligned}$$

One final example:

$$\frac{21}{35} = \frac{\square}{5}$$

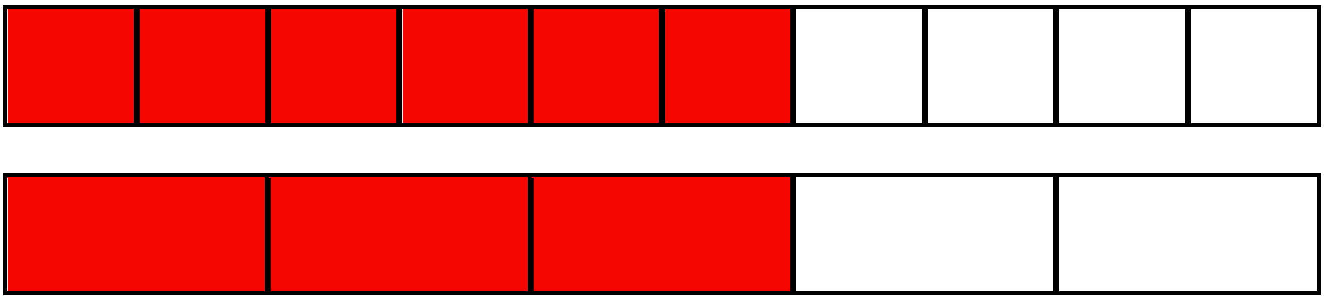
We see that to move from 35 to 5 we need to divide by 7, so we must do the same to the top:

$$\begin{aligned} \frac{21}{35} &= \frac{21 \div 7}{35 \div 7} \\ &= \frac{3}{5}. \end{aligned}$$

SIMPLIFYING

Often after doing some calculations with fractions you are left with a fraction containing large numbers. At this point you would be expected to simplify. This is because $\frac{6}{7}$ is much easier to look at and understand than $\frac{1512}{1764}$!

Let's simplify $\frac{6}{10}$. Using our visualisation we see that:



So we can see that $\frac{6}{10}$ is equivalent to $\frac{3}{5}$. Alternatively we can see that both 6 and 10 are divisible by 2 (in fact 2 is the only number, apart from 1, that we can divide both 6 and 10 by), so we would write

$$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}.$$

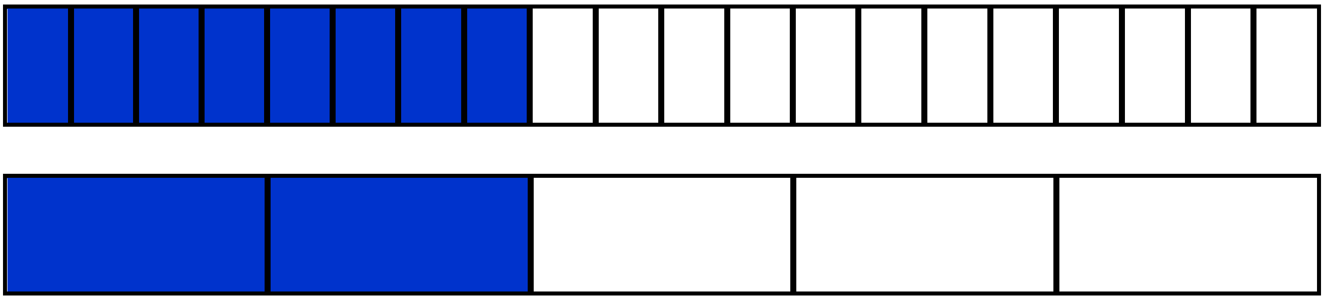
Next consider $\frac{8}{20}$. We can simplify this fraction in either one or two steps. Most of the time one step will be quicker and is the better method to use, however in some cases, when the numbers in the fraction are large, it will be better to simplify in two (or more) steps.

To simplify in one step, we look for the largest number that both numerator and denominator are divisible by (this is called the *highest common factor* or *greatest common divisor*). For 8 and 20 this number is 4. So our solutions looks like:

$$\frac{8}{20} = \frac{8 \div 4}{20 \div 4} = \frac{2}{5}.$$

To use multiple steps we just look for any number that divides both numerator and denominator and keep doing this until 1 is the only number that will divide both (i.e. 1 is the only factor the numerator and denominator have in common). So for instance we could have divided both 8 and 20 by 2 to get the fraction $\frac{4}{10}$, then divided both 4 and 10 by 2 again to obtain $\frac{2}{5}$.

Using the visualisation technique we can check that these fractions are indeed equivalent.



Let's consider a hard example such as $\frac{1512}{1764}$. It is not at all obvious what the largest number that divides both 1512 and 1764 is, so we will need to use steps. Firstly you might notice that both numbers are even, so both are divisible by 2!

$$\frac{1512}{1764} = \frac{1512 \div 2}{1764 \div 2} = \frac{756}{882}.$$

Both numbers are still even so we can again divide by 2.

$$\frac{756}{882} = \frac{756 \div 2}{882 \div 2} = \frac{378}{441}$$

Hmm, the numbers are still large but now one of them is odd, so we can't divide by 2! As there is no obvious number that goes into both, let's try 3 (if you know any divisibility checks they would be useful here).

$$\frac{378}{441} = \frac{378 \div 3}{441 \div 3} = \frac{126}{147}.$$

The division worked out, so 3 was a correct number to use. The numbers are a lot smaller now, although not small enough so that the largest divisor is obvious. However, you may notice that they are again divisible by 3:

$$\frac{126}{147} = \frac{126 \div 3}{147 \div 3} = \frac{42}{49}.$$

These numbers are much nicer, and we can now see that they are both divisible by 7, so:

$$\frac{1512}{1764} = \frac{42}{49} = \frac{42 \div 7}{49 \div 7} = \frac{6}{7}.$$

And we have finally simplified it.

Note: we could use our workings to find the greatest common divisor. We divided by 2, then 2 again, then 3, 3 again, and finally 7. If we multiply all these numbers together we get $2 \times 2 \times 3 \times 3 \times 7 = 252$, which is therefore the greatest common divisor of 1512 and 1764.

Here are some for you to try (nothing as hard as that last one). You can check your answers with the solutions at the end of this resource.

EXERCISES

5. Fill in the blanks in each of the following sets of equivalent fractions:

(a) $\frac{5}{8} = \frac{\square}{16}$ (b) $\frac{9}{12} = \frac{3}{\square}$ (c) $\frac{7}{10} = \frac{\square}{100}$ (d) $\frac{35}{100} = \frac{7}{\square}$

6. Simplify the following fractions by putting them into their lowest form:

(a) $\frac{90}{100}$ (b) $\frac{24}{32}$ (c) $\frac{25}{75}$ (d) $\frac{12}{36}$

If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

1. Proper
2. Improper, $1\frac{1}{4}$
3. Proper
4. Improper, $2\frac{5}{6}$

5. (a) $\frac{5}{8}=\frac{\square}{16}$: 8 has been multiplied by 2 to get 16, so:

$$\frac{5}{8}=\frac{5\times 2}{8\times 2}=\frac{10}{16}$$

(b) $\frac{9}{12}=\frac{3}{\square}$: 9 has been divided by 3 to get 3, so:

$$\frac{9}{12}=\frac{9\div 3}{12\div 3}=\frac{3}{4}$$

(c) $\frac{7}{10}=\frac{\square}{100}$: 10 has been multiplied by 10 to get 100, so:

$$\frac{7}{10}=\frac{7\times 10}{10\times 10}=\frac{70}{100}$$

(d) $\frac{35}{100}=\frac{7}{\square}$: 35 has been divided by 5 to get 7, so:

$$\frac{35}{100}=\frac{35\div 5}{100\div 5}=\frac{7}{20}$$

6. (a) $\frac{90}{100}$: both 90 and 100 are divisible by 10, so:

$$\frac{90}{100}=\frac{90\div 10}{100\div 10}=\frac{9}{10}$$

(b) $\frac{24}{32}$: both 24 and 32 are divisible by 8, so:

$$\frac{24}{32}=\frac{24\div 8}{32\div 8}=\frac{3}{4}$$

(c) $\frac{25}{75}$: both 25 and 75 are divisible by 25, so:

$$\frac{25}{75}=\frac{25\div 25}{75\div 25}=\frac{1}{3}$$

(d) $\frac{12}{36}$: both 12 and 36 are divisible by 12, so:

$$\frac{12}{36}=\frac{12\div 12}{36\div 12}=\frac{1}{3}$$