# Surds

## Definition and Manipulation

### INTRODUCTION

Before defining a surd, let's think about squares and other powers. (If you need to review powers, you might like to have a look at the documents *Calculations – Order of Operations in Mathematics* and *Powers and Logarithms – 1. Powers, Indices, Exponents*).

We know that $3^2 = 9$, because $3 \times 3 = 9$. But what about trying to reverse this procedure? That is, we might want to know what number multiplies by itself to give $9$. We need a new notation for this and the symbol to use is "$\sqrt{\ }$". In the case where we are trying to work out what number multiplied by itself gives $9$, we use $\sqrt{9}$. This means that $\sqrt{9}$ is $3$, because $3 \times 3 = 9$.

In the same way, $\sqrt{4}$ is $2$, because $2 \times 2 = 4$; and $\sqrt{25}$ is $5$ because $5 \times 5 = 25$.

But what do we mean by $\sqrt{2}$? We don't know straight away what number multiplies by itself to give $2$, so we might have to take a guess. While we're trying to work it out though, let's just the notation $\sqrt{2}$ to represent the number which multiplies by itself to give $2$.

### WE CANNOT DETERMINE THE EXACT VALUE OF $\sqrt{2}$

If you multiply $1$ by $1$, you get $1$. If you multiply $2$ by $2$, you get $4$. So we know that somewhere between $1$ and $2$ there must be a number that you can multiply by itself and get $2$. Try $1.5$. But $1.5 \times 1.5 = 2.25$, which is too big. So try $1.4$, but $1.4 \times 1.4 = 1.96$, too small, but a bit closer than $1.5$ gave us. Try $1.42$. But $1.42 \times 1.42 = 2.0164$, just a bit too big, so try $1.419$, say… (I'm getting sick of this!) Let's use the calculator – mine gives $\sqrt{2} = 1.414213562$ but a computer can give many more decimal places than this, and the truth is that there is no end of or pattern to the decimal expansion for this number. This means that $\sqrt{2}$ is *irrational* or more specifically a *surd*.

### A SURD IS A ROOT FOR WHICH WE CANNOT DETERMINE THE EXACT VALUE

We can only hope to approximate surds using estimates. That is, a surd gives us a way of writing that exact value – for example, $\sqrt{2}$ is the notation for the exact value of the square root of $2$, that is, the number which multiplied by itself gives $2$. Surds don't have to be <u>square</u> roots –they could be a cubed root, fourth root, and so on. For example $\sqrt[3]{5}$ is a surd because, again trying different numbers, we find $1 \times 1 \times 1 = 1$, $2 \times 2 \times 2 = 8$, $3 \times 3 \times 3 = 27$, but we don't know any number that we can do this with and end up with $5$. The calculator gives $1.709975947\ldots$

We have just seen that $\sqrt{2}$ is a surd. $\sqrt{3}$ is also a surd, but $\sqrt{4}$ is <u>not</u> a surd because $2 \times 2 = 4$, so we can easily determine an exact value for $\sqrt{4}$.

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$ are surds. But, as we saw, $\sqrt{4}$ is not a surd. What about $\sqrt{9}$? As we also saw before, $\sqrt{9}$ is not a surd either because we know that $3^2 = 3 \times 3 = 9$, so $\sqrt{9} = 3$.

Thinking about $\sqrt{2}$, perhaps the best definition is this:

$$\sqrt{2} \times \sqrt{2} = 2$$

That is, $\sqrt{2}$ is the number that we multiply (or "times") by itself to get $2$. In the same way, $\sqrt{3} \times \sqrt{3} = 3$, and $\sqrt{7} \times \sqrt{7} = 7$.



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Here are some for you to try:

$\sqrt{5} \times \sqrt{5} = \_\_$ $\sqrt{11} \times \sqrt{11} = \_\_;$ $\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10} = \_\_$

Answers: 5, 11, 10

In general, for ANY positive number – let's call it $a$, for example, so that $a$ represents any positive number – then

$$\sqrt{a} \times \sqrt{a} = a$$

$$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$$

And so on.

This works for any number (as long as it is positive), so $\sqrt{10.9} \times \sqrt{10.9} = 10.9$ and $\sqrt[4]{23} \times \sqrt[4]{23} \times \sqrt[4]{23} \times \sqrt[4]{23} = 23$. Here, we don't even need to know what $\sqrt{10.9}$ or $\sqrt[4]{23}$ are! We just need to understand what the notation means.

## CALCULATING WITH SURDS

Now, let's look at multiplying and dividing with surds.

**Multiplication**

We know that, for example, $\sqrt{2} \times \sqrt{2} = 2$. But what about $\sqrt{2} \times \sqrt{3}$? We can actually multiply these surds together to get $\sqrt{6}$. (You can check this result on the calculator if you like.) In general, for any positive numbers $a$ and $b$,

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$$

So:

— $\sqrt{3} \times \sqrt{5} = \sqrt{15}$

— $\sqrt{6} \times \sqrt{5} = \sqrt{30}$

— $4\sqrt{2} \times 3\sqrt{5} = 12\sqrt{10}$ (because we multiply the 4 by 3 and the surds by each other)

— $\sqrt{5} \times \sqrt{2} \times \sqrt{5} = 5\sqrt{2}$ (because we can multiply the $\sqrt{5}$ by $\sqrt{5}$ to get 5 and multiply that by $\sqrt{2}$ to get $5\sqrt{2}$)

— $4\sqrt{2} \times 3\sqrt{2} = 12 \times 2 = 24$.

**Division**

We can also combine surds when dividing them.

For example, $\sqrt{3} \div \sqrt{5} = \frac{\sqrt{3}}{\sqrt{5}} = \sqrt{\frac{3}{5}} = \sqrt{0.6}$

Or, going the other way, $\sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$

In general, for any positive numbers $a$ and $b$,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

## EXERCISES

**Simplify the following:**

1. $3\sqrt{5} \times 2\sqrt{3}$

2. $3\sqrt{5} \times 2\sqrt{5}$

3. $3\sqrt{5} \times 2\sqrt{2} \times \sqrt{3}$

4. $\left(\sqrt{3}\right)^2$

5. $\dfrac{\sqrt{20}}{\sqrt{5}}$

6. $\sqrt{\dfrac{25}{9}}$

### Addition and Subtraction

We can only add or subtract LIKE surds, that is, surds that have the same number (or expression) under the $\sqrt{\phantom{x}}$ sign. So, for example, we can add $\sqrt{5}$ and $3\sqrt{5}$ (that is, $\sqrt{5} + 3\sqrt{5}$) to get $4\sqrt{5}$, or we can subtract $\sqrt{5}$ from $3\sqrt{5}$ (that is, $3\sqrt{5} - \sqrt{5}$) to get $2\sqrt{5}$, but we can<u>not</u> add or subtract $\sqrt{5}$ and $\sqrt{2}$ because they are UNLIKE surds.

Here are some for you to try: Work out which of these expressions can be simplified, that is, are there any like surds?

$3\sqrt{5} + 2\sqrt{7}$; $4\sqrt{5} - 2\sqrt{3}$; $3\sqrt{2} + 5\sqrt{2}$; $9\sqrt{7} - 4\sqrt{6}$

Answer: Only the third expression can be simplified and its value is $8\sqrt{2}$ .

### Simplifying Surds

There are ways to simplify some surds so that they can become like other surds! For example, let's look at $\sqrt{8}$. Now, $\sqrt{8} = \sqrt{4 \times 2}$, and from the section on multiplying surds (above), we can see that $\sqrt{4 \times 2}$ can also be written as $\sqrt{4} \times \sqrt{2}$. As we know that $\sqrt{4} = 2$, $\sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2}$, or, for short, $2\sqrt{2}$. So $\sqrt{8} = 2\sqrt{2}$.

In the same way, we could simplify $\sqrt{63}$ as follows:

$$\begin{aligned}\sqrt{63} &= \sqrt{9 \times 7}\\ &= \sqrt{9} \times \sqrt{7}\\ &= 3 \times \sqrt{7}\\ &= 3\sqrt{7}\end{aligned}$$

This method of simplifying surds allows us to add or subtract some surds that at first look unlike.

*Examples:*

Simplify $\sqrt{12} + \sqrt{27}$

$$\begin{aligned}\sqrt{12} + \sqrt{27} &= \sqrt{4 \times 3} + \sqrt{9 \times 3}\\ &= 2\sqrt{3} + 3\sqrt{3}\\ &= 5\sqrt{3}\end{aligned}$$

Simplify $\sqrt{12} + \sqrt{27} - \sqrt{75}$

$$\begin{aligned}\sqrt{12} + \sqrt{27} - \sqrt{75} &= \sqrt{4 \times 3} + \sqrt{9 \times 3} - \sqrt{25 \times 3}\\ &= 2\sqrt{3} + 3\sqrt{3} - 5\sqrt{3}\\ &= 0\end{aligned}$$

Simplify $2\sqrt{128} - 3\sqrt{98} + \sqrt{72}$

$$\begin{aligned}2\sqrt{128} - 3\sqrt{98} + \sqrt{72} &= 2\sqrt{64 \times 2} - 3\sqrt{49 \times 2} + \sqrt{36 \times 2}\\ &= 2 \times 8\sqrt{2} - 3 \times 7\sqrt{2} + 6\sqrt{2}\\ &= 16\sqrt{2} - 21\sqrt{2} + 6\sqrt{2}\\ &= \sqrt{2}\end{aligned}$$

Notice that to simplify each surd above we have used the largest factor that has a square root. For $\sqrt{72}$, for example, we used $72 = 36 \times 2$, however, we could have obtained the same result if we had used $9 \times 8$, as follows:

$$\begin{aligned}\sqrt{72} &= \sqrt{9 \times 8} \\ &= 3\sqrt{8} \\ &= 3\sqrt{4 \times 2} \\ &= 3\ \times 2\sqrt{2} \\ &= 6\sqrt{2}\end{aligned}$$

This method took a little longer than the other one, but it worked!

Here are some for you to try. You can check your answers with the solutions at the end of this resource.

## EXERCISES

**Simplify these surds:**

6. $\sqrt{54}$ 7. $\sqrt{50}$ 8. $\sqrt{32}$ 9. $\sqrt{200}$

**Simplify these expressions:**

10. $\sqrt{75} + \sqrt{48}$ 11. $\sqrt{108} - \sqrt{12}$ 12. $4\sqrt{150} - 3\sqrt{54}$

13. $2\sqrt{5}\ \times 4\sqrt{5}$ 14. $\left(3\sqrt{2}\right)^2$ 15. $\sqrt{\frac{25}{16}}$ 16. $\frac{\sqrt{54}}{\sqrt{6}}$

# MORE MULTIPLYING AND DIVIDING WITH SURDS

**Expanding Brackets**

*Examples*

Simplify $\left(3\sqrt{2} + 1\right)\left(\sqrt{2} - 5\right)$

$$\begin{aligned}\left(3\sqrt{2} + 1\right)\left(\sqrt{2} - 5\right) &= 3\sqrt{2} \times \sqrt{2} + 3\sqrt{2} \times -5 + 1 \times \sqrt{2} + 1 \times -5 \\ &= 3 \times 2 - 15\sqrt{2} + \sqrt{2} - 5 \\ &= 6 - 5 - 14\sqrt{2} \\ &= 1 - 14\sqrt{2}\end{aligned}$$

Simplify $\left(3 - \sqrt{2}\right)^2$

$$\begin{aligned}(3 - \sqrt{2}\,)^2 &= \left(3 - \sqrt{2}\right)\left(3 - \sqrt{2}\right) \\ &= 3 \times 3 + 3\ \times -\sqrt{2} + \ -\sqrt{2} \times 3 + \ -\sqrt{2} \times -\sqrt{2} \\ &= 9 - 3\sqrt{2} - 3\sqrt{2} + 2 \\ &= 11 - 6\sqrt{2}\end{aligned}$$

Alternatively, use formula: $(a-b)^2 = a^2 - 2ab + b^2$

## EXERCISES

**Simplify the following:**

17. $(4+\sqrt{3})(2-\sqrt{3})$

18. $(2\sqrt{2}-5)(3\sqrt{2}-1)$

19. $(1+\sqrt{2})^2$

20. $(3\sqrt{5}-2\sqrt{3})^2$

### Conjugate Surds

An important fact with surds is that when we multiply a surd by its *conjugate*, we obtain a rational number*. Examples of conjugate pairs are $\sqrt{6}-\sqrt{2}$ and $\sqrt{6}+\sqrt{2}$; $2\sqrt{5}+3$ and $2\sqrt{5}-3$ and so on.

Let's see what happens when we multiply $\sqrt{6}-\sqrt{2}$ and $\sqrt{6}+\sqrt{2}$:

$$(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2}) = 6+\sqrt{12}-\sqrt{12}-2 = 4,$$

which is a rational number. Note that this is actually using the *Difference of Two Squares* formula:

$$(a-b)(a+b) = a^2 - b^2.$$

Let's try $(2\sqrt{5}+3)(2\sqrt{5}-3)$:

$$\begin{aligned}(2\sqrt{5}+3)(2\sqrt{5}-3) &= 20-6\sqrt{5}+6\sqrt{5}-9 \\ &= 11,\end{aligned}$$

which is also a rational number.

This is very handy when a fraction has a surd in the denominator – see later.

## EXERCISES

**Multiply each of the following by its conjugate:**

21. $5+\sqrt{3}$

22. $\sqrt{3}-1$

23. $2\sqrt{3}+3\sqrt{2}$

### Surds in the Denominator

If we multiply any number by 1, we do not change the value of that number. This allows us to change denominators of numbers to whatever we like. In this case we consider $\frac{2}{\sqrt{3}}$ and we multiply it by 1, but it doesn't look like 1:

$$\begin{aligned}\frac{2}{\sqrt{3}} &= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{2\sqrt{3}}{3}\end{aligned}$$

Note firstly that $\frac{\sqrt{3}}{\sqrt{3}}$ is actually 1 in disguise! Notice also that now the denominator in the answer is a rational number and is much easier to calculate with. This process is called rationalising the denominator.

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* A rational number is one that can be expressed as a fraction. For example, $\frac{4}{5}$ is rational, 3.5 is rational – because you can express it as $\frac{7}{2}$ and any whole number, or integer, is rational because you can express them as a fraction with 1 in the denominator, but surds can't be expressed that way, because their decimal value does not terminate or repeat and continues forever so we can't write them as a fraction.

*Examples*

Rationalise the denominator of the fraction $\frac{4}{\sqrt{2}-1}$

We multiply by the conjugate of the denominator, because, as we have seen, multiplying conjugates gives a rational result.

So,

$$\begin{aligned}\frac{4}{\sqrt{2}-1} &= \frac{4}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\\ &= \frac{4\left(\sqrt{2}+1\right)}{2-1}\\ &= 4\left(\sqrt{2}+1\right)\end{aligned}$$

Rationalise the denominator of the fraction $\frac{3+2\sqrt{5}}{3-2\sqrt{5}}$

$$\begin{aligned}\frac{3+2\sqrt{5}}{3-2\sqrt{5}} &= \frac{3+2\sqrt{5}}{3-2\sqrt{5}}\times\frac{3+2\sqrt{5}}{3+2\sqrt{5}}\\ &= \frac{9+6\sqrt{5}+6\sqrt{5}+20}{9-20}\\ &= \frac{29+12\sqrt{5}}{-11}\\ &= -\left(\frac{29+12\sqrt{5}}{11}\right)\end{aligned}$$

## EXERCISES

**Rationalise the denominator in each of the following fractions:**

24. $\frac{5}{\sqrt{3}}$

25. $\frac{4}{2\sqrt{2}}$

26. $\frac{9}{\sqrt{3}+1}$

27. $\frac{\sqrt{2}+5}{3-\sqrt{2}}$

28. $\frac{1+3\sqrt{2}}{2\sqrt{2}+3}$

*If you need help with any of the maths covered in this resource (or any other maths topics), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.*

## SOLUTIONS TO EXERCISES

1. $3\sqrt{5} \times 2\sqrt{3} = 6\sqrt{15}$

2. $3\sqrt{5} \times 2\sqrt{5} = 6 \times 5 = 30$

3. $3\sqrt{5} \times 2\sqrt{2} \times \sqrt{3} = 6\sqrt{30}$

4. $\left(\sqrt{3}\right)^2 = 3$

5. $\dfrac{\sqrt{20}}{\sqrt{5}} = \sqrt{\dfrac{20}{5}} = \sqrt{4} = 2$

6. $\sqrt{\dfrac{25}{9}} = \dfrac{\sqrt{25}}{\sqrt{9}} = \dfrac{5}{3}$

7. $\sqrt{54} = \sqrt{9 \times 6} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$

8. $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

8. $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

9. $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

10. $\sqrt{75} + \sqrt{48} = \sqrt{25 \times 3} + \sqrt{16 \times 3} = 5\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}$

11. $\sqrt{108} - \sqrt{12} = \sqrt{36 \times 3} - \sqrt{4 \times 3} = 6\sqrt{3} - 2\sqrt{3} = 4\sqrt{3}$

12. $4\sqrt{150} - 3\sqrt{54} = 4 \times \sqrt{25 \times 6} - 3 \times \sqrt{9 \times 6}$

$= 4 \times 5\sqrt{6} - 3 \times 3\sqrt{6}$

$= 20\sqrt{6} - 9\sqrt{6}$

$= 11\sqrt{6}$

13. $2\sqrt{5} \times 4\sqrt{5} = 8 \times 5 = 40$

14. $\left(3\sqrt{2}\right)^2 = 3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18$

15. $\sqrt{\dfrac{25}{16}} = \dfrac{\sqrt{25}}{\sqrt{16}} = \dfrac{5}{4}$

16. $\dfrac{\sqrt{54}}{\sqrt{6}} = \sqrt{\dfrac{54}{6}} = \sqrt{9} = 3$

17. $\left(4 + \sqrt{3}\right)\left(2 - \sqrt{3}\right) = 4 \times 2 + 4 \times -\sqrt{3} + \sqrt{3} \times 2 + \sqrt{3} \times -\sqrt{3}$

$= 8 - 4\sqrt{3} + 2\sqrt{3} - 3$

$= 5 - 2\sqrt{3}$

18. $\left(2\sqrt{2} - 5\right)\left(3\sqrt{2} - 1\right) = 2\sqrt{2} \times 3\sqrt{2} + 2\sqrt{2} \times -1 - 5 \times 3\sqrt{2} - 5 \times -1$

$= 12 - 2\sqrt{2} - 15\sqrt{2} + 5$

$= 17 - 17\sqrt{2}$

$= 17\left(1 - \sqrt{2}\right)$

19. $\left(1+\sqrt{2}\right)^2 = \left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right) = 1+\sqrt{2}+\sqrt{2}+2 = 3+2\sqrt{2}$

20. $\left(3\sqrt{5}-2\sqrt{3}\right)^2 = \left(3\sqrt{5}-2\sqrt{3}\right)\left(3\sqrt{5}-2\sqrt{3}\right) = 45-6\sqrt{15}-6\sqrt{15}+12 = 57-12\sqrt{15}$

21. $\left(5+\sqrt{3}\right)\left(5-\sqrt{3}\right) = 25-5\sqrt{3}+5\sqrt{3}-3 = 22$

22. $\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right) = 3-1 = 2$

23. $\left(2\sqrt{3}+3\sqrt{2}\right)\left(2\sqrt{3}-3\sqrt{2}\right) = 12-18 = -6$

24. $\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$

25. Note, you have a choice with this one. You can multiply by either $\frac{\sqrt{2}}{\sqrt{2}}$ or $\frac{2\sqrt{2}}{2\sqrt{2}}$. So:

$$\frac{4}{2\sqrt{2}} = \frac{4}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{4} = \sqrt{2}$$

Or

$$\frac{4}{2\sqrt{2}} = \frac{4}{2\sqrt{2}} \times \frac{2\sqrt{2}}{2\sqrt{2}} = \frac{8\sqrt{2}}{8} = \sqrt{2}$$

26.

$$\frac{9}{\sqrt{3}+1} = \frac{9}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{9\left(\sqrt{3}-1\right)}{3-1} = \frac{9\left(\sqrt{3}-1\right)}{2}$$

27.

$$\frac{\sqrt{2}+5}{3-\sqrt{2}} = \frac{\sqrt{2}+5}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3\sqrt{2}+2+15+5\sqrt{2}}{9-2} = \frac{8\sqrt{2}+17}{7}$$

28.

$$\frac{1+3\sqrt{2}}{2\sqrt{2}+3} = \frac{1+3\sqrt{2}}{2\sqrt{2}+3} \times \frac{2\sqrt{2}-3}{2\sqrt{2}-3} = \frac{2\sqrt{2}-3+12-9\sqrt{2}}{8-9} = \frac{-7\sqrt{2}+9}{-1} = 7\sqrt{2}-9$$