Powers and Logarithms

1. Powers, Indices, Exponents

INTRODUCTION

You may have come across the terms *powers*, *indices*, *exponents*, and *logarithms*. But what do they mean?

The terms *power(s)*, *index (indices)*, *exponent(s)* in mathematics are actually interchangeable. All of them refer to that little number written above and to the right of another number, such as the 2 in 5^2 or the 3 in 4^3. Some of those little numbers (written as superscripts) have special names. You are probably familiar with *squaring* and *cubing* a number. The term *logarithm(s)* is different to the other terms but has a strong connection to them, we will cover them in *Powers and Logarithms – 2. Logarithms*. But let's start at the beginning!

POWERS

At some point in our calculations we may have to evaluate 3×3, however rather than writing out both those 3s, we can use a shorthand notation: 3^2. The superscript $\blacksquare^2$ tells us that the 3 is to be multiplied by itself, and we would get the answer 9. Note: you have seen this type of shorthand before, in fact multiplication is the shorthand for repeated addition, i.e. instead of writing $3 + 3$ we write 2×3. Be careful not to confuse the two though, $3^2 = 3 \times 3 = 9$, while $2 \times 3 = 3 + 3 = 6$.

In the example 3^2, 3 is called the *base* and 2 is called the *power* (or *index* or *exponent*). We'll use *power* from now on, but remember that we can just as easily write *index* or *exponent*. 3^2 is read as *three raised to the power of two*, or simply *three to the power two*. More commonly, when the power is 2, we use the word *squared*, so we can also read this as *three squared*. No matter which way we express it, 3^2 will always mean 3×3 and give the answer 9.

A further example: 4^3. This is read as *four raised to the power of three*, or *four to the power three*, or *four cubed*. It means $4 \times 4 \times 4$, that is three fours multiplied together, and will give the result 64 because $4 \times 4 = 16$ and $16 \times 4 = 64$. The *base* in this case is 4 and the *power* is 3.

By the way, the powers 2 and 3 are the only ones that have special names. So, for example, 5^4 is read as *five [raised] to the power [of] four*, or *five [raised] to the fourth [power]*. Note, that a number raised to the power 1 is just the number itself, e.g. $3^1 = 3$ and $752^1 = 752$, etc.

Another example: 2^4 (read *two to the power four*) is $2 \times 2 \times 2 \times 2$, which makes 16. Here, the *base* is 2 and the *power* is 4. (Note how efficient the notation is – we don't have to write out all those 2s!) Understanding the terms *base* and *power* is important as these terms will be used later when defining *logarithms*.

EXERCISES

Find the value of each of the following. Also, for each question, work out which numbers represent the base and the power.

1. 2^3 2. 3^4 3. 10^2 4. 5^3

You can check these results on your calculator and also explanations are provided at the end of this resource. If you are not sure how to use your calculator, you can have a look at *Calculators – Getting to Know Your Scientific Calculator*.

RAISING A NEGATIVE NUMBER TO A POWER

Let's say we have to raise -3 to the power 2. This MUST be written as $(-3)^2$. The reason for this is that we need to multiply -3×-3. If we write -3^2, without the brackets, this implies that we square the 3 first (because of order of

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operations), then put a minus sign in front of the answer! (For example this is what you do when evaluating $15 - 3^2$, say, which is the same as $15 - 9$ and gives 6.) The correct answer to raising -3 to the power 2 is 9. If you square any real number, positive or negative, you will ALWAYS get a positive result.

What about $(-2)^3$? This means $-2 \times -2 \times -2$, and gives the result -8 (because $-2 \times -2 = 4$, note the intermediate result is positive, and $4 \times -2 = -8$, the next multiplication makes it negative again.) Notice this time, when we *cube* a negative number, we obtain a negative result. Here are some for you to try.

EXERCISES

Find the value of each of the following.

5. $(-4)^2$ 6. $(-3)^4$ 7. $10^3 - 5^3$ 8. $10^3 + (-5)^3$ 9. $10^2 - 4^2$

Again you can check these results on your calculator and the answers are explained at the end.

POWERS OF FRACTIONS

Let's briefly look at raising a fraction to a power. For example $\left(\frac{3}{4}\right)^2$ this is the same as $\frac{3}{4} \times \frac{3}{4}$, which gives $\frac{9}{16}$. Note how this is the same as $\frac{3^2}{4^2}$, when raising a fraction to a power we can always raise the numerator and denominator separately. (Remember, when multiplying fractions, multiply across numerators and across denominators. If you are not sure how to, you can refer to *Fractions – 2. Multiplication and Division*.)

OPERATIONS WITH POWERS: MULTIPLICATION AND DIVISION

Let's bring in a little bit of algebra here. Don't run away though, algebra just lets us generalise what happens to numbers. So, for example a^2 just means $a \times a$ (no different to what we have seen before), where a is the *base*, and 2 is the *power*. The only difference is that a can represent any number.

There are some shortcuts to working out calculations with powers. Say we want to calculate $a^2 \times a^3$. We could write this out longhand and obtain $(a \times a) \times (a \times a \times a)$, but since the only operation in this expression is multiplication we don't need the brackets. That is, it is the same as $a \times a \times a \times a \times a$, which is a^5. Notice that the power, 5, is also the result of adding the powers 2 and 3. This happens in every case. So, to multiply two powers of the same base, just add the powers.

This is our first general rule for operations with powers. We can use letters for the powers as well, but remember the letters simply stand for the general case and you can use the rule every time you recognise it.

$$a^p \times a^q = a^{p+q}$$

Example: Simplify $2^5 \times 2^4$.

Using the rule: $2^5 \times 2^4 = 2^{5+4} = 2^9$.

(Longer method: $2^5 \times 2^4 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$.)

Note that the base numbers must be the same for this rule to work.

Now, we've seen that when you multiply the same base number raised to powers, that you can actually add the powers. So it follows that if you are dividing the same base number raised to powers then you would…

Let's see what happens using longhand notation:

$$3^6 \div 3^2 = \frac{\cancel{3} \times \cancel{3} \times 3 \times 3 \times 3 \times 3}{\cancel{3} \times \cancel{3}}$$
$$= \frac{3 \times 3 \times 3 \times 3}{1}$$
$$= 3^4$$

So we have subtracted the powers! This gives us our second general rule for operations with powers:

$$a^p \div a^q = a^{p-q}$$

EXERCISES

Simplify the following:

10. $2^3 \times 2^5$

11. $3^8 \div 3^4$

12. $5^4 \times 5^3 \div 5^2$

OPERATIONS WITH POWERS: POWERS OF POWERS (SUPERPOWERS?)

What happens when we raise a number that is already raised to a power to another power? This sounds confusing so let's look at an example: $(2^3)^4$. This would mean $2^3 \times 2^3 \times 2^3 \times 2^3$ and if we add all the powers we would end up with 2^{12}. (And if we wrote out what each 2^3 is we get $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ and we still get 2^{12}!) Notice we could have obtained the same result for the power if we had multiplied the 3 and the 4. This is our next general rule:

$$(a^p)^q = a^{p \times q} = a^{pq}$$

EXERCISES

Write each of the following as a number raised to a single power:

13. $(3^2)^4$

14. $(2^5)^2$

15. $(4^1)^3$

16. $(10^2)^3$

OPERATIONS WITH POWERS: POWERS OF ZERO AND NEGATIVE POWERS

Consider what would happen if we have $5^2 \div 5^2$. By first simplifying the $5^2 = 25$ we would obtain $25 \div 25 = 1$. But what about if we use our division rule to simplify?

Well using the subtraction of powers we have

$$5^2 \div 5^2 = 5^{2-2} = 5^0.$$

So, this means that 5^0 is the same as 1. This goes for any number, so we can generalise again and write:

$$a^0 = 1$$

EXERCISES

17. 10^0

18. 384^0

19. 4×3^0

20. $5^3 \div 5^0$

21. $(3 \times 5)^0$

22. $4a^0$

Let's now go further than zero and into negative powers. Say we have to calculate $3^4 \div 3^5$. Using the subtraction of powers, 3^{4-5}, we end up with 3^{-1}. Whereas using division we get:

$$3^4 \div 3^5 = \frac{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3} \times 3} = \frac{1}{3}$$

This means that 3^{-1} is the same as $\frac{1}{3}$. If we had $3^2 \div 3^6$, we would end up with 3^{-4} or $\frac{1}{3^4}$, and this leads us to our next rule:

$$a^{-p} = \frac{1}{a^p}$$

One last example before you get to try some: $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$.

EXERCISES

23. 2^{-5}

24. 7^{-3}

25. $4^2 \div 4^5$

OPERATIONS WITH POWERS: FRACTIONAL POWERS

Suppose we multiply $\sqrt{2} \times \sqrt{2}$. We must get 2, because that is what $\sqrt{2}$ means. In the same way, $\sqrt{3} \times \sqrt{3} = 3$ and $\sqrt{185} \times \sqrt{185} = 185$. Or writing it as algebra we have $\sqrt{a} \times \sqrt{a} = a$, where a represents any number.

But what about the following calculation?

$$\begin{aligned} 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} &= 2^{\frac{1}{2}+\frac{1}{2}} \\ &= 2^1 \\ &= 2 \end{aligned}$$

Because we are multiplying $2^{\frac{1}{2}}$ by itself, and ending up with 2, just as we did when we multiplied $\sqrt{2}$ by itself, and ended up with 2, it follows, since are both positive, that actually $\sqrt{2}$ is the same as $2^{\frac{1}{2}}$. And $\sqrt{3}$ is the same as $3^{\frac{1}{2}}$ and so on. So we can write:

$$a^{\frac{1}{2}} = \sqrt{a}$$

By the same logic we have $a^{\frac{1}{3}} = \sqrt[3]{a}$ (the cube root of a), and $a^{\frac{1}{4}} = \sqrt[4]{a}$ (the fourth root of a). For example $\sqrt[3]{8} = 2$ because we look for the number, which when cubed gives 8. And so, $8^{\frac{1}{3}} = 2$ for the same reason!

What about $8^{\frac{2}{3}}$ though? We are going to use the power of a power rule but we can think of this in two ways. Firstly we note that since $8^{\frac{2}{3}} = 8^{2\times\frac{1}{3}}$ it is the same as $(8^2)^{\frac{1}{3}}$, which means we look for $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$. Alternatively we could say $8^{\frac{2}{3}} = 8^{\frac{1}{3}\times 2} = \left(8^{\frac{1}{3}}\right)^2$, which means we have $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$. Either way works (however the simpler numbers are probably easier).

Another example: $16^{\frac{3}{2}}$. Let's split it as $\left(16^{\frac{1}{2}}\right)^3$ as this will result in simpler/smaller numbers. $16^{\frac{1}{2}} = 4$ and $4^3 = 64$, so $16^{\frac{2}{3}} = 64$.

(Notice we always use improper fractions for powers and do not use mixed numbers, because of the confusion mixed numbers create in regards to multiplication, hence we use $\frac{3}{2}$ instead of $1\frac{1}{2}$.)

Again we can generalise are write the rule:

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

EXERCISES

26. $25^{\frac{1}{2}}$

27. $81^{\frac{1}{4}}$

28. $49^{\frac{3}{2}}$

29. $125^{\frac{2}{3}}$

You can check your answers on your calculator. If you are not sure how to do this please refer to *Calculators – Getting to Know Your Scientific Calculator*.

If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

1. 2^3 is 8 (it is $2 \times 2 \times 2$)

2. $3^4 = 3 \times 3 \times 3 \times 3 = 81$

3. $10^2 = 10 \times 10 = 100$

4. $5^3 = 5 \times 5 \times 5 = 125$

5. $(-4)^2 = -4 \times -4 = 16$

6. $(-3)^4 = -3 \times -3 \times -3 \times -3 = 81$

7. $10^3 - 5^3 = 10 \times 10 \times 10 - 5 \times 5 \times 5 = 1000 - 125 = 875$

8. $10^3 + (-5)^3 = 10 \times 10 \times 10 + -5 \times -5 \times -5 = 1000 + -125 = 875$

9. $10^2 - 4^2 = 10 \times 10 - 4 \times 4 = 100 - 16 = 84$

10. $2^3 \times 2^5 = 2^{3+5} = 2^8$

11. $3^8 \div 3^4 = 3^{8-4} = 3^4$

12. $5^4 \times 5^3 \div 5^2 = 5^{4+3-2} = 5^5$ (or you can do it in steps: $5^{4+3} \div 5^2 = 5^7 \div 5^2 = 5^{7-2} = 5^5$)

13. $(3^2)^4 = 3^{2\times 4} = 3^8$

14. $(2^5)^2 = 2^{5\times 2} = 2^{10}$

15. $(4^1)^3 = 4^{1\times 3} = 4^3$

16. $(10^2)^3 = 10^{2\times 3} = 10^6$

17. $10^0 = 1$

18. $384^0 = 1$

19. $4 \times 3^0 = 4 \times 1 = 4$

20. $5^3 \div 5^0 = 5^{3-0} = 5^3$ (or $5^3 \div 1 = 5^3$)

21. $(3 \times 5)^0 = 1$ (it does not matter what the inside equals)

22. $4a^0 = 4 \times a^0 = 4 \times 1 = 4$

23. $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

24. $7^{-3} = \frac{1}{7^3} = \frac{1}{343}$

25. $4^2 \div 4^5 = 4^{2-5} = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

26. $25^{\frac{1}{2}} = \sqrt{25} = 5$

27. $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

28. $49^{\frac{3}{2}} = \left(\sqrt{49}\right)^3 = 7^3 = 343$

29. $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$