Algebra

7. Factorising Pairs of Terms

Before looking at factorising, you might like to have a look at *Algebra – 6. Expanding Algebraic Expressions*.

INTRODUCTION

Firstly let's think about the word *factor*. In English, we think of factors as things that might contribute to some result, such as when making a decision, we consider all the factors that we need to – when deciding whether to go out somewhere, we look at the weather, other things we need to do, how are we going to get there, if we have enough money, and so on. All these are "factors" that contribute to our making the decision to go out.

In maths, when talking about numbers, *factors* are generally integers (whole numbers) that divide into a number without any remainder. When a pair of factors are multiplied together they form the number, that is, if you like, they contribute to making that number. So, for example, the factors of 6 are 1, 6, 2 and 3, because when we multiply 1 and 6 we obtain 6 and when we multiply 2 and 3, we obtain 6 (while negative numbers are considered factors, we often don't write them down so as to avoid repetition). It doesn't matter what order we list factors in, as long as we work them all out! So, we could have written 3, 1, 2, 6 as the factors of 6, but it is usually easier to start from 1 (and the number itself) and work up from there. However, it is often useful to list them in an increasing order.

Example

Work out all the factors of 10.

Answer: 1, 10, 2, 5

Often a number, such as 24, has many factors but many other numbers have no factors except 1 and themselves. The factors of 24 are 1, 24, 2, 12, 3, 8, 4, 6, but the factors of 29 are only 1 and 29. (29 is called a *prime number*, because of its factors. A prime number has only two factors, 1 and itself. All other whole numbers greater than 1 are called *composite numbers*. The first prime number is 2 and after that comes 3, 5, 7, and so on. 2 is also the only even number that is prime, since 2 is a factor of all other even numbers. We don't include 1 as prime because it doesn't have two factors.)

Now, we need to think about *algebraic factors*, in other words factors that contribute to, or multiply together, to make up algebraic expressions. a^2, for example, is made up of $a \times a$ and so the only factors of a^2 are 1, a^2, and a. $3a$ is made up of $3 \times a$, so its factors are 1, $3a$, 3, and a. xy is made up of $x \times y$, so its factors are 1, xy, x and y.

FACTORISING

Writing an algebraic expression as a product of its algebraic factors is called *factorising*. Usually when factorising, we are considering more than one term. From the handout on *Algebra – 6. Expanding Algebraic Expressions*, we learned that the result of multiplying $2(x+1)$ is $2x+2$. Now, we look at going backwards, and working out the factors of $2x+2$. We do this by thinking of what terms could have possibly contributed to making up both $2x$ and 2.

List the factors of $2x$ and of 2:

- For $2x$, we have 1, $2x$, 2 and x.
- For 2, we have 1 and 2.

So, the common factors of $2x$ and 2 are 1 and 2. The highest common factor (the one with the largest value) is 2, so we now work out what we multiply 2 by to make both $2x$ and 2. $2 \times x$ is $2x$ and 2×1 is 2 so the factors of $2x+2$ are $2 \times x + 2 \times 1$, which can be put together as 2 times both x and 1, that is $2(x+1)$.

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Now let's factorise $3a + 12$.

- List the factors of $3a$: $1, 3a, 3, a$.
- List the factors of 12: $1, 12, 2, 6, 3, 4$.

The common factors of both $3a$ and 12 are 1 and 3, the highest is 3. So we get $3a + 12 = 3 \times a + 3 \times 4$ and that gives $3a + 12 = 3(a + 4)$.

What about $3a - 12$? Well, the numbers are still the same, so we only need to consider signs when we are multiplying, and we can see that $3a - 12 = 3 \times a - 3 \times 4$ and so $3a - 12 = 3(a - 4)$

Now let's try $10x^2 + 6x$

- List all factors of $10x^2$: $1, 10x^2, 2, 5x^2, 10, x^2, x, 10x, 2x, 5x$
- List all factors of $6x$: $1, 6x, 2, 3x, 3, 2x, 6, x$

Let's think of a quicker way! The highest common factor of the numbers 10 and 6 is 2 (apart from 1, 2 is the only number that divides exactly into both 10 and 6), and the highest common factor of x^2 and x is x. Putting them together gives the highest common factor of $10x^2$ and $6x$ as $2x$.

So,

$$10x^2 + 6x = 2x(5x + 3)$$

Separating the numbers from the pronumerals (the terms in x) is generally easier to work out the highest common factor than listing all the factors of each algebraic term.

Try some for yourself. You can check your answers with the solution at the end of this resource.

EXERCISES

Factorise the following expressions. Check your answers by expanding the factors.

1. $2a + 10$ 2. $10x - 25$ 3. $15y - 21$ 4. $8 + 30p$

5. $x^2 + 4x$ 6. $3m^2 + 5m$ 7. $abc + ac$ 8. $a^2b + ab^2$

EXPRESSIONS WITH MORE THAN TWO TERMS

So far we've only been dealing with pairs of terms. We can have any number of terms which we can factorise. For example, let's try to factorise $8x^2y + 4xy^2 + 16xy$. Firstly, look at the numbers. The highest common factor of 8, 4, and 16 is 4. The highest common factor of x^2y, xy^2, and xy is xy. So we get:

$$8x^2y + 4xy^2 + 16xy = 4xy(2x + y + 4).$$

You should always check factorising by expanding back. Using the table method:

$\mathbf{4xy(2x + y + 4)}$

$\times$	$2x$	y	4
$4xy$	$8x^2y$	$4xy^2$	$16xy$

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So our factors are correct, that is, $8x^2y + 4xy^2 + 16xy = 4xy(2x + y + 4)$.

Pairs and Grouping

Often, when there are 4 terms in an algebraic expression, they have been made up of 2 pairs. For example, we can factorise $px + qx + py + qy$ by grouping the four terms into the two pairs of $px + qx$ and $py + qy$ and obtain:

$$px + qx + py + qy = x(p + q) + y(p + q).$$

Now, the next step involves realising that we now have the compound term, $p + q$ being multiplied by both x and y, which means $p + q$ is a common factor to both terms. So, we obtain the following:

$$\begin{aligned} px + qx + py + qy &= x(p + q) + y(p + q) \\ &= (p + q)(x + y) \end{aligned}$$

(Check this by expanding).

In the same way, we can factorise $x^3 - 3x^2 + 2x - 6$ by grouping the first two terms and the last two terms, as follows:

$$x^3 - 3x^2 + 2x - 6 = x^2(x - 3) + 2(x - 3)$$

Now, $(x - 3)$ is common to both terms and we get:

$$x^3 - 3x^2 + 2x - 6 = (x^2 + 2)(x - 3)$$

Remember, the order in which we write the brackets doesn't matter – we could have correctly put $(x - 3)(x^2 + 2)$ because when we multiply out the brackets we get the same result – you should check this!

Tricky minuses!

Example 1.

Factorise $ac - ad - bc + bd$:

Taking care with signs and grouping in pairs gives

$$ac - ad - bc + bd = a(c - d) - b(c - d)$$

Now we see that $c - d$ is multiplied by both a and $-b$, so we get

$$ac - ad - bc + bd = (c - d)(a - b)$$

Check by expanding. Using the table method:

$$(\boldsymbol{c} - \boldsymbol{d})(\boldsymbol{a} - \boldsymbol{b})$$

$\times$	c	$-d$
a	ac	$-ad$
$-b$	$-bc$	bd

We can see we have all the terms in the original expression $ac - ad - bc + bd$, so the factors are correct.

Example 2.

Factorise $ax - ay + by - bx$:

Again, group the pairs so that you have common factors:

$$ax - ay + by - bx = a(x - y) + b(y - x)$$

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Oh dear! The $(x-y)$ and $(y-x)$ are the reverse of each other. Not to worry, just as $5-3$ is the opposite of $3-5$, $(x-y)$ is the opposite of $(y-x)$ and we can write $(y-x)$ as $-(x-y)$. We can check this by expanding $-(x-y)$: we would get $-x+y$, which is the same as $y-x$. So,

$$\begin{aligned} ax-ay+by-bx &= a(x-y)+b(y-x) \\ &= a(x-y)-b(x-y) \end{aligned}$$

Now we have our common factor and can combine this into $(a-b)(x-y)$, and to check we would just expand out again.

Here are some examples for you to try. Again, you can check your results with the solutions at the end.

EXERCISES

Factorise each of the following algebraic expressions:

9. $ap+aq+pr+qr$

10. $7x+14y+ax+2ay$

11. $2ab-3bc+4a-6c$

12. $x^3+5x^2+2x+10$

13. $8a-80b-a^2+10ab$

14. xy^2+xz-y^2p-zp

15. $2-2a+ba-b$

16. $15y-5+2x-6xy$

If you need help with any of the maths covered in this resource (or any other maths topics), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

1. $2a + 10 = 2(a + 5)$

2. $10x - 25 = 5(2x - 5)$

3. $15y - 21 = 3(5y - 7)$

4. $8 + 30p = 2(4 + 15p)$

5. $x^2 + 4x = x(x + 4)$

6. $3m^2 + 5m = m(3m + 5)$

7. $abc + ac = ac(b + 1)$

8. $a^2b + ab^2 = ab(a + b)$

9. $ap + aq + pr + qr = a(p + q) + r(p + q)$
$= (a + r)(p + q)$

10. $7x + 14y + ax + 2ay = 7(x + 2y) + a(x + 2y)$
$= (7 + a)(x + 2y)$

11. $2ab - 3bc + 4a - 6c = b(2a - 3c) + 2(2a - 3c)$
$= (b + 2)(2a - 3c)$

12. $x^3 + 5x^2 + 2x + 10 = x^2(x + 5) + 2(x + 5)$
$= (x^2 + 2)(x + 5)$

13. $8a - 80b - a^2 + 10ab = 8(a - 10b) - a(a - 10b)$
$= (8 - a)(a - 10b)$

14. $xy^2 + xz - y^2p - zp = x(y^2 + z) - p(y^2 + z)$
$= (x - p)(y^2 + z)$

15. $2 - 2a + ba - b = 2(1 - a) + b(a - 1)$
$= 2(1 - a) - b(1 - a)$
$= (2 - b)(1 - a)$

16. $15y - 5 + 2x - 6xy = 5(3y - 1) + 2x(1 - 3y)$
$= 5(3y - 1) - 2x(3y - 1)$
$= (5 - 2x)(3y - 1)$