Algebra

3. Solving Equations

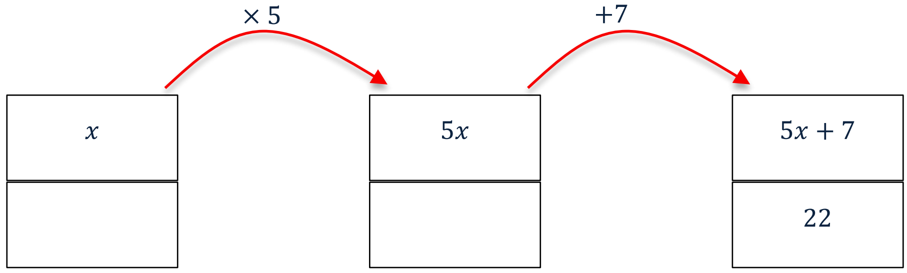
An *algebraic equation* is where two algebraic expressions are equal to each other. Finding the solution of the equation means finding the value(s) of the *variable* which makes the equation true. There may be one or more, or even *no* solutions to the equation.

An example of an equation which has no solution using the set of real numbers is $x = \sqrt{-9}$. Try this on your calculator. We can't find the square root of a negative number because no matter what real number we square we always end up with a positive result.

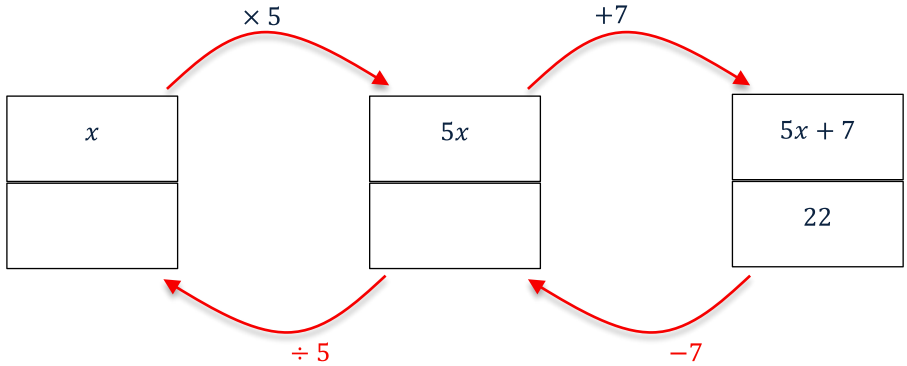
One of the easiest ways in which to solve an equation is to use the idea of *backtracking*. With this, you start by working forwards, working out what has been done to the variable step by step, then working backwards to undo that procedure.

Here is an example: Find the value of x if $5x + 7 = 22$.

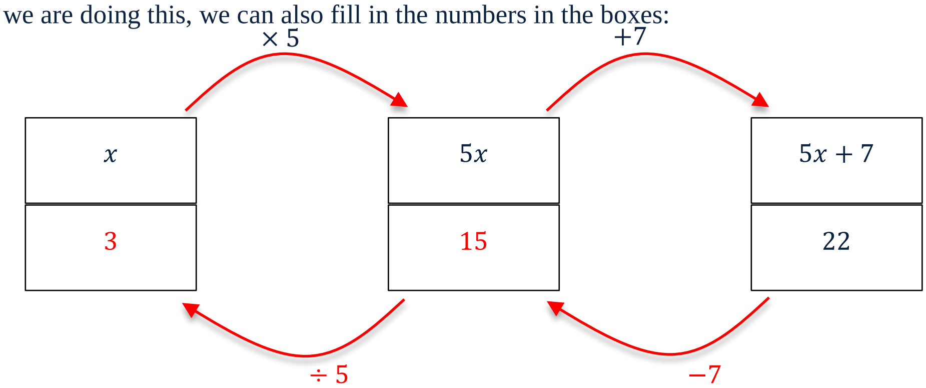
Firstly, we map what has been done to the variable (in this case, x):



Now we work out what to do to go backwards, starting from the right-hand box.



While we are doing this, we can also fill in the numbers in the boxes:



This tells us that $x = 3$. Last, we check that this value fits the original equation:

$x = 3$ gives $5x + 7 = 5 \times 3 + 7 = 15 + 7 = 22$, which agrees with the original equation.

We can also solve this equation with the *balance both sides* technique (which is a different version of the backtracking method):

$$5x + 7 = 22$$

The balance both sides technique works because if we do the same thing to things that are equal they remain equal. Think of a pair of weighing scales that are initially balanced, if you add a 5 kg weight to both sides of the scale they will still be balanced. In this case we subtract 7 from both sides (to undo the $+7$):

$$5x + 7 - 7 = 22 - 7$$

So,

$$5x = 15.$$

Note, this is the same as the middle boxes in the backtracking method.

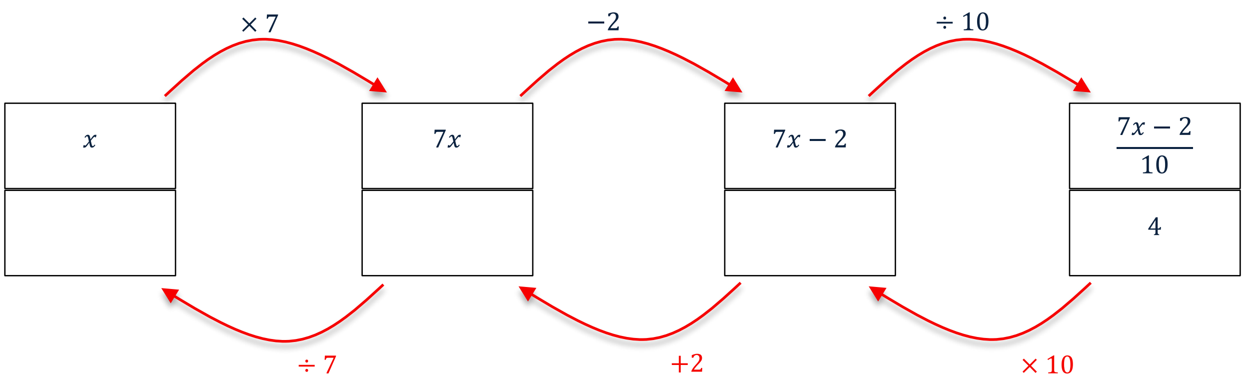
Now we divide both sides by 5 (to undo the $\times 5$):

$$\frac{5x}{5} = \frac{15}{5}$$

$$x = 3.$$

Try the backtracking technique on a more difficult equation:

$$\frac{7x - 2}{10} = 4$$



Now the balancing both sides technique:

$$\frac{7x - 2}{10} = 4$$

Multiply both sides by 10 (to undo the $\div 10$)

$$\left(\frac{7x - 2}{10}\right) \times 10 = 4 \times 10$$

$$7x - 2 = 40$$

Add 2 to both sides (to undo the -2)

$$7x - 2 + 2 = 40 + 2$$

$$7x = 42$$

Divide both sides by 7 (to undo the $\times 7$)

$$\frac{7x}{7} = \frac{42}{7}$$

$$x = 6.$$

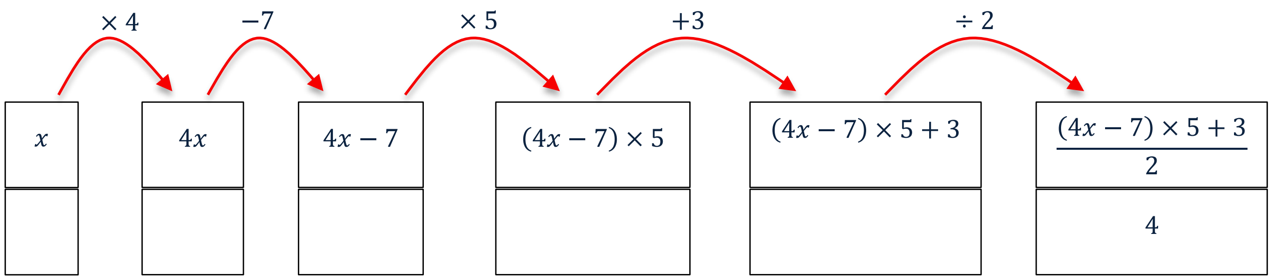
Check $x = 6$ works in the original equation. We use LHS and RHS to stand for the left-hand side and right-hand side of the equation, respectively. We need to confirm that when $x = 6$ the LHS equals the RHS

$$\begin{aligned} LHS &= \frac{7x - 2}{10} \\ &= \frac{7 \times 6 - 2}{10} \\ &= \frac{42 - 2}{10} \\ &= \frac{40}{10} \\ &= 4 = RHS \end{aligned}$$

So $x = 6$ is the correct solution to the equation.

A final example: Solve $\frac{(4x-7)\times 5+3}{2} = 4$

Using backtracking – you can fill in the blanks and work backwards



Answer: $x = 2$

The idea with solving equations using either technique is to work out what has been done to the variable (often x) and then to undo those steps in the reverse order. This is analogous to getting dressed and undressed. In the morning you put on a shirt then put on a jacket, similar to multiplying x by 5 then adding 7, while at night you do the reverse, you take off the jacket then take off the shirt, similar to subtracting 7 then dividing by 5.

EXERCISES

Solve for x

1. $(2x - 1) \times 3 = 21$

2. $5(x - 2) - 3 = 42$

3. $\frac{6x+8}{4} = 11$

4. $\frac{11(3x+5)}{2} = 44$

INVERSE OPERATIONS

Unfortunately, the backtracking method is only valid for equations when the variable only appears once in the equation. However, the general techniques of "undoing" each step work in every situation. Before we go on, check that you know the inverse procedure (the "undoing" technique) for each of the following operations:

OPERATION	INVERSE OPERATION
$+$	
$-$	
$\times$	
$\div$	
Squaring ($^\wedge 2$)	
Raising to the power a ($^\wedge a$)	

Answers: $-$, $+$, $\div$, $\times$, Square rooting ($\sqrt{\square}$), Raising to the power $\frac{1}{a}$

Example:

Solve: $\sqrt{x+1} = 3$

What has been done to the variable (x) here?

Firstly, 1 has been added, then the square root has been taken off the result. We need to undo these operations in the reverse order (you can't take your shirt off before your jacket). So, firstly, square both sides to undo the square root:

$$\left(\sqrt{x+1}\right)^2 = 3^2$$
$$x + 1 = 9$$

Take away 1 from both sides:

$$x + 1 - 1 = 9 - 1$$
$$x = 8$$

Check the solution:

$$\begin{aligned} LHS &= \sqrt{x+1} \\ &= \sqrt{8+1} \\ &= \sqrt{9} \\ &= 3 = RHS \end{aligned}$$

So $x = 8$ is the correct solution.

WHERE THE VARIABLE APPEARS MORE THAN ONCE

When the variable appears more than once in an equation the idea is to combine them so that the variable only appears once. This may not be possible, in these cases other techniques need to be employed. Often combining variables will involve moving them from one side of the equation to the other.

Example 1

Solve $5x - 6 = 3x + 2$

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To combine the terms with x in them we need to have them on the same side of the equation. This can be achieved by subtracting $3x$ from both sides:

$$5x - 6 - 3x = 3x + 2 - 3x$$

$$2x - 6 = 2$$

Now that the variable only appears once we can proceed as we had before, either using backtracking or balancing both sides. We will use balancing both sides here, so we add 6 to both sides

$$2x = 8$$

$$x = 4$$

Check the solution. This time we need to substitute into both LHS and RHS:

$$\begin{aligned} LHS &= 5x - 6 \\ &= 5 \times 4 - 6 \\ &= 20 - 6 \\ &= 14 \end{aligned} \qquad \begin{aligned} RHS &= 3x + 2 \\ &= 3 \times 4 + 2 \\ &= 12 + 2 \\ &= 14 \end{aligned}$$

So $x = 4$ is the correct solution since $LHS = RHS$.

Example 2

Solve $4x + 3 = 8x - 5$

This time we subtract $4x$ from both sides:

$$4x + 3 - 4x = 8x - 5 - 4x$$

$$3 = 4x - 5$$

Now, add 5 to both sides

$$3 + 5 = 4x - 5 + 5$$

$$8 = 4x$$

$$2 = x$$

This is the same as $x = 2$. Check the solution:

$$\begin{aligned} LHS &= 4x + 3 \\ &= 4 \times 2 + 3 \\ &= 8 + 3 \\ &= 11 \end{aligned} \qquad \begin{aligned} RHS &= 8x - 5 \\ &= 8 \times 2 - 5 \\ &= 16 - 5 \\ &= 11 \end{aligned}$$

So $x = 2$ is the correct solution since $LHS = RHS$.

Example 3

Solve $5(2x + 1) = 3(4x - 1)$

Here it is easiest to expand both sides (multiply out the brackets) first. This is because the variable is trapped within the brackets, and we need to let it out before we can do anything.

$$10x + 5 = 12x - 3$$

Now we can subtract $10x$ from both sides

$$10x + 5 - 10x = 12x - 3 - 10x$$

$$5 = 2x - 3$$

Add 3 to both sides

$$5 + 3 = 2x - 3 + 3$$

$$8 = 2x$$

Divide both sides by 2

$$\frac{8}{2} = \frac{2x}{2}$$

$$4 = x$$

So $x = 4$.

Check the solution.

$$\begin{aligned} LHS &= 5(2x + 1) \\ &= 5(2 \times 4 + 1) \\ &= 5(8 + 1) \\ &= 5(9) \\ &= 45 \end{aligned}$$

$$\begin{aligned} RHS &= 3(4x - 1) \\ &= 3(4 \times 4 - 1) \\ &= 3(16 - 1) \\ &= 3(15) \\ &= 45 \end{aligned}$$

So $x = 4$ is the correct solution since $LHS = RHS$.

EXERCISES

Solve each equation for x:

5. $3x + 5 = 5x + 1$

6. $4x - 1 = 3(2x - 5)$

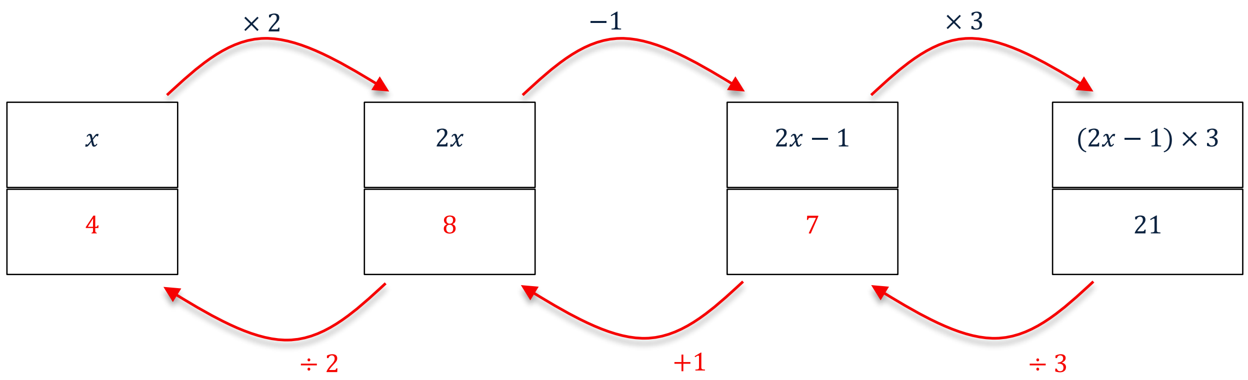
7. $12 - 2x = x - 3$

If you need help with any of the maths covered in this resource (or any other maths topics), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

1. $(2x - 1) \times 3 = 21$

Backtracking method:



Balancing both sides technique:

$$(2x - 1) \times 3 = 21$$

Divide both sides by 3:

$$\frac{(2x - 1) \times 3}{3} = \frac{21}{3}$$

$$2x - 1 = 7$$

Add 1 to both sides:

$$2x - 1 + 1 = 7 + 1$$

$$2x = 8$$

Divide both sides by 2:

$$\frac{2x}{2} = \frac{8}{2}$$

$$\underline{x = 4}$$

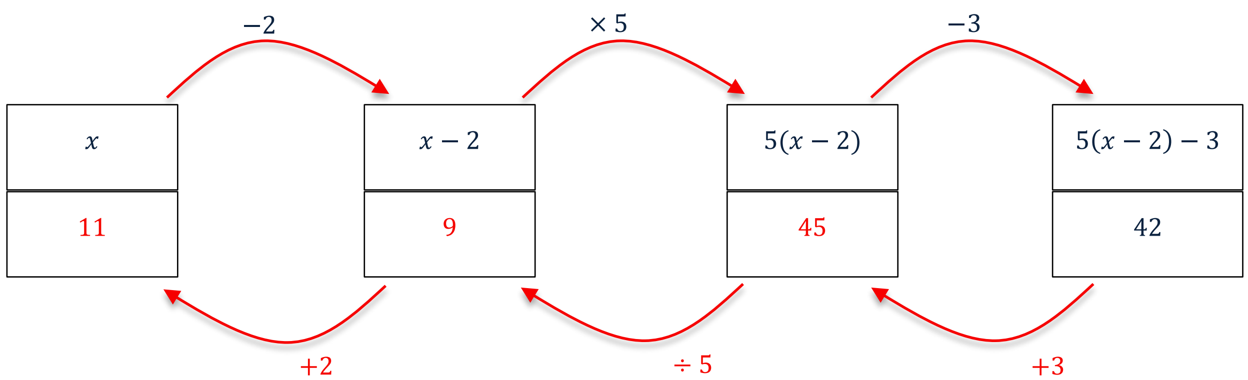
Check solution:

$$\begin{aligned} LHS &= (2x - 1) \times 3 \\ &= (2 \times 4 - 1) \times 3 \\ &= (8 - 1) \times 3 \\ &= 7 \times 3 \\ &= 21 = RHS \end{aligned}$$

So $x = 4$ is the correct solution.

2. $5(x - 2) - 3 = 42$

Backtracking method:



Balancing both sides technique:

$$5(x-2)-3=42$$

Add 3 to both sides:

$$5(x-2)-3+3=42+3$$
$$5(x-2)=45$$

Divide both sides by 5:

$$\frac{5(x-2)}{5}=\frac{45}{5}$$
$$x-2=9$$

Add 2 to both sides:

$$\underline{x=11}$$

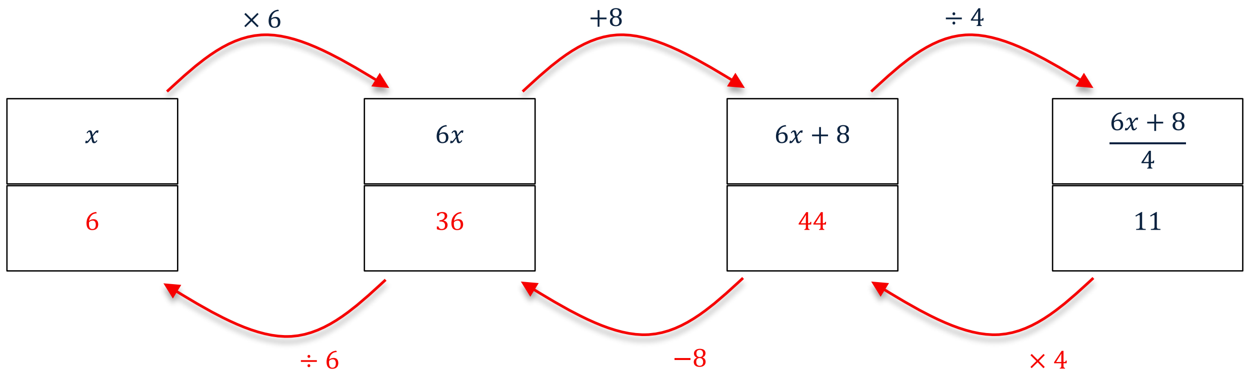
Check solution:

$$\begin{aligned} LHS &= 5(x-2)-3 \\ &= 5(11-2)-3 \\ &= 5\times 9-3 \\ &= 45-3 \\ &= 42 = RHS \end{aligned}$$

So $x=11$ is the correct solution.

3. $\frac{6x+8}{4}=11$

Backtracking method:



Balancing both sides technique:

$$\frac{6x+8}{4}=11$$

Multiply both sides by 4:

$$\frac{6x+8}{4}\times 4=11\times 4$$
$$6x+8=44$$

Subtract 8 from both sides:

$$6x+8-8=44-8$$
$$6x=36$$

Divide both sides by 6:

$$\underline{x=6}$$

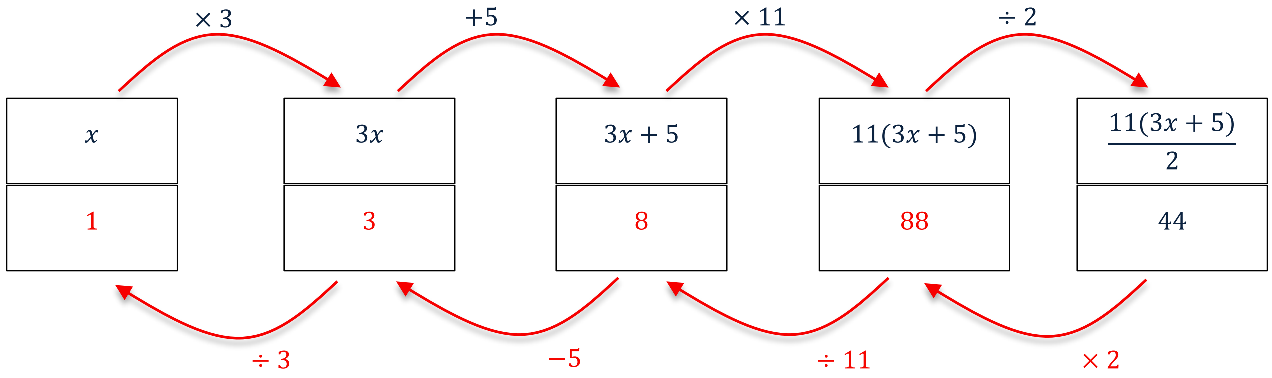
Check solution:

$$\begin{aligned} LHS &= \frac{6x+8}{4} \\ &= \frac{36+8}{4} \\ &= \frac{44}{4} \\ &= 11 = RHS \end{aligned}$$

So $x=6$ is the correct solution.

4. $\frac{11(3x+5)}{2} = 44$

Backtracking method:



Balancing both sides technique:

$$\frac{11(3x + 5)}{2} = 44$$

Multiply both sides by 2:

$$\frac{11(3x + 5)}{2} \times 2 = 44 \times 2$$

$$11(3x + 5) = 88$$

Divide both sides by 11:

$$\frac{11(3x + 5)}{11} = \frac{88}{11}$$

$$3x + 5 = 8$$

Subtract 5 from both sides:

$$3x + 5 - 5 = 8 - 5$$

$$3x = 3$$

Divide both sides by 3:

$$\underline{x = 1}$$

Check solution:

$$LHS = \frac{11(3x + 5)}{2}$$

$$= \frac{11(3 \times 1 + 5)}{2}$$

$$= \frac{11(3 + 5)}{2}$$

$$= \frac{11 \times 8}{2}$$

$$= \frac{88}{2}$$

$$= 44 = RHS$$

So $x = 1$ is the correct solution.

5. $3x + 5 = 5x + 1$

Subtract $3x$ from both sides:

$$3x + 5 - 3x = 5x + 1 - 3x$$

$$5 = 2x + 1$$

Subtract 1 from both sides:

$$5 - 1 = 2x + 1 - 1$$

$$4 = 2x$$

Divide both sides by 2:

$$2 = x$$

This is the same as $x = 2$. Check the solution:

$$\begin{aligned} LHS &= 3x + 5 \\ &= 3 \times 2 + 5 \\ &= 6 + 5 \\ &= 11 \end{aligned} \qquad \begin{aligned} RHS &= 5x + 1 \\ &= 5 \times 2 + 1 \\ &= 10 + 1 \\ &= 11 \end{aligned}$$

So $x = 2$ is the correct solution since $LHS = RHS$.

6. $4x - 1 = 3(2x - 5)$

Expand the RHS:

$$4x - 1 = 6x - 15$$

Subtract $4x$ from both sides:

$$4x - 1 - 4x = 6x - 15 - 4x$$
$$-1 = 2x - 15$$

Add 15 from both sides:

$$-1 + 15 = 2x - 15 + 15$$
$$14 = 2x$$

Divide both sides by 2 and reverse the equation:

$$x = 7$$

Check the solution:

$$\begin{aligned} LHS &= 4x - 1 \\ &= 4 \times 7 - 1 \\ &= 28 - 1 \\ &= 27 \end{aligned} \qquad \begin{aligned} RHS &= 3(2x - 5) \\ &= 3(2 \times 7 - 5) \\ &= 3(14 - 5) \\ &= 3 \times 9 \\ &= 27 \end{aligned}$$

So $x = 7$ is the correct solution since $LHS = RHS$.

7. $12 - 2x = x - 3$

Add $2x$ to both sides:

$$12 = 3x - 3$$

Add 3 to both sides

$$15 = 3x$$

Divide both sides by 3 and reverse the equation:

$$x = 5$$

Check the solution:

$$\begin{aligned} LHS &= 12 - 2 \times 5 \\ &= 2 \end{aligned} \qquad \begin{aligned} RHS &= 5 - 3 \\ &= 2 \end{aligned}$$

So $x = 5$ is the correct solution since $LHS = RHS$.