Volume and Surface Area

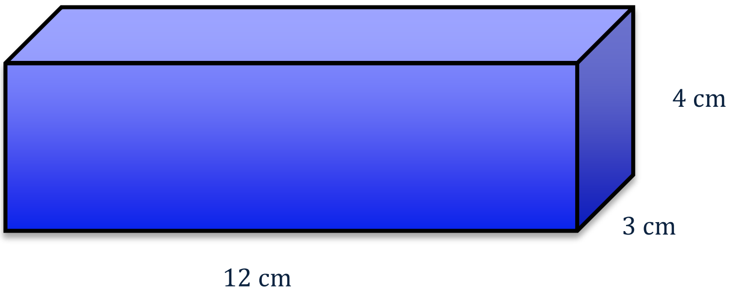
2. Surface Areas

INTRODUCTION

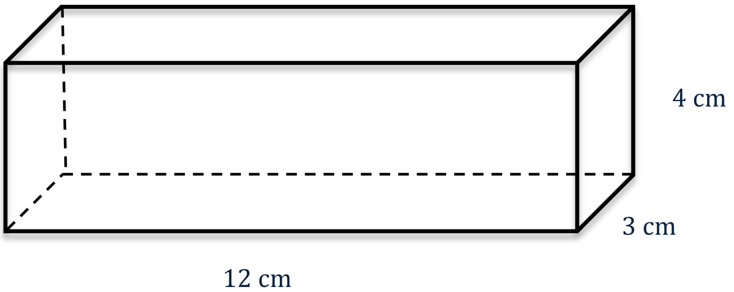
If you need to know more about plane shapes, areas, perimeters, solids or volumes of solids, please refer to *Area and Perimeter – 1. Areas of Plane Shapes*, *Area and Perimeter – 2. Perimeters of Plane Shapes*, and *Volume and Surface Area – 1. Volumes of Solids*. You may also need to review Pythagoras' theorem – if so, please refer to *Pythagoras' Theorem*.

The *surface area* of a 3D shape is "the total area of the outside" of the shape (De Klerk, 2007, p. 129). Therefore, to work out the surface area of a shape, you need to work out the area of each face and add these areas together. Before we calculate the surface area of an example, remember that since we are measuring an area, we will be measuring in square units.

Example: calculate the surface area of the following rectangular prism:



It may become clearer what we have to do if we draw in the edges that are blocked from sight, as this allows us to see <u>all</u> faces.



We see that there are six faces, all of them rectangles, so each of their areas is $A = l \times w$. For each of the top and bottom faces the area is $12\text{ cm} \times 3\text{ cm} = 36\text{ cm}^2$. For each of the left and right faces the area is $3\text{ cm} \times 4\text{ cm} = 12\text{ cm}^2$. For each of the front and back faces the area is $12\text{ cm} \times 4\text{ cm} = 48\text{ cm}^2$. So the total surface are is:

$$\begin{aligned} SA &= 2 \times 36\text{ cm}^2 + 2 \times 12\text{ cm}^2 + 2 \times 48\text{ cm}^2 \\ &= 192\text{ cm}^2. \end{aligned}$$

For any prism where the base is a polygon, i.e. a triangle, a rectangle, a pentagon, etc., the faces will consist of two copies of the base and some *rectangles*. This allows an easy calculation of the surface area using the same method as above. However when we have a circular prism (i.e. a cylinder), things are slightly more complicated.

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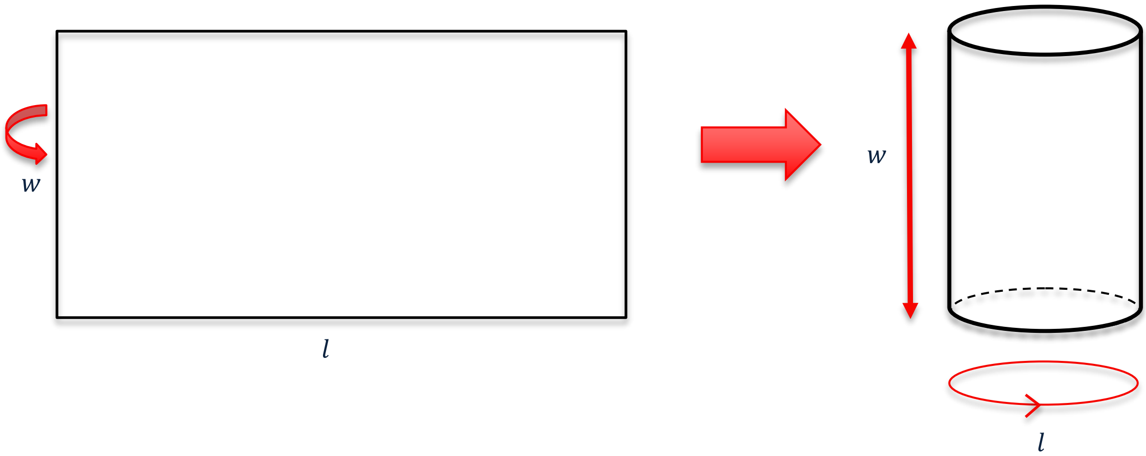
SURFACE AREA OF A CYLINDER

In the case of a cylinder, its faces consist of up to two circles (depending on if the cylinder is open or not) and a curved surface.

We can work out the areas of the circles by using the formula $A = \pi r^2$, but what about the curved surface? Consider taking sheet of paper and joining two opposite sides together. We have just created a (hollow) cylinder with the curved surface being our sheet of paper. Importantly, we have not changed the area! So the area of this curved shape is the length of our sheet of paper times the width of the sheet of paper (since the sheet of paper is a rectangle). In cylinder form it is easy to see that width of the piece of paper is the "height" of the cylinder, while the length of the piece of paper has become the *circumference* of the circle (or vice versa, depending on which way you rolled the cylinder, but this won't matter in the end).



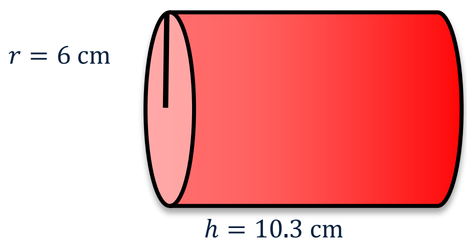
Now the circumference of a cylinder is given by $l = 2\pi r$, and since we know that $h = w$, we obtain a formula for the area:

$$A = 2\pi r \times h$$

For our hollow cylinder, this is precisely its surface area. We call such a cylinder *open at both ends*. We could also have a cylinder that is open at one end and closed at the other, or one that is closed at both ends. The total surface area in each case is given below:

Cylinder Type	Surface Area
Open at Both Ends	$A = 2\pi r \times h$
Closed at One End	$A = 2\pi r \times h + \pi r^2$
Closed at Both Ends	$A = 2\pi r \times h + 2\pi r^2$

Example: Find the surface area of the following closed cylinder:



Since it is closed at both ends we use the formula $A = 2\pi r \times h + 2\pi r^2$. Therefore the surface area is:

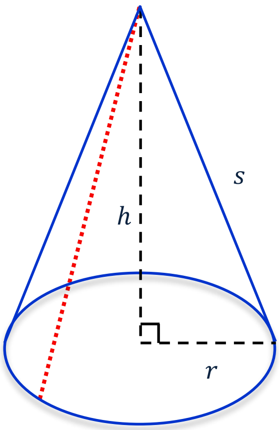
$$A = 2 \times \pi \times 6 \text{ cm} \times 10.3 \text{ cm} + 2 \times \pi \times 6^2 \text{ cm}^2 = 614.495523 \ldots \text{cm}^2$$

So the surface area is approximately 614.496 cm^2.

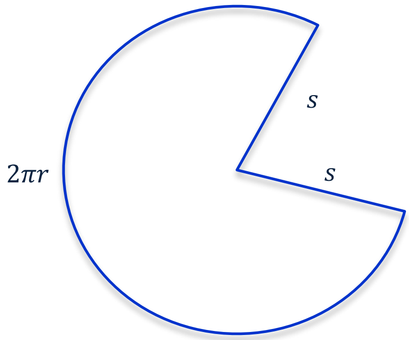
SURFACE AREA OF A CONE

Any pyramid has a polygon base, so its faces consist of the base and some triangles. The surface area is then the area of the base added to the area of the triangles. However, when we have a cone, we again end up with needing to calculate the area of a curved surface. This curved surface is more complicated to calculate.

Notice that we can to move from an open cylinder to a plane rectangle by doing the reverse of our steps in the previous section. That is we cut along the height of the cylinder and "unfold" our surface. Let's do the same thing for an open cone and cut along the red dotted line:



This time if we unfold our sheet, we no longer obtain a rectangle but instead part of a circle! Try this yourself. Start by rolling a piece of paper into a cone, then cut away any overlapping bits and cut it so the base is flat. Now unroll it. You should obtain something that looks like Pac-man:



The straight edges of our sector come from where we cut our surface and the curved line comes from the base of our cone, so its length is the same as the circumference of the base, that is $2\pi r$. But how can we work out the radius of the sector? Well this is the length we cut along, so it is the length of the slant, s.

Now it might be that we know what this length is. In this case, to work out the area of this sector, we calculate what fraction of a circle our sector is. We can do this by calculating the fraction of the circumference that we have:

$$\text{Fraction of circle} = \frac{2\pi r}{2\pi s} = \frac{r}{s}$$

This is also the fraction of the circle's area that our sector takes up! Therefore

$$A = \frac{r}{s} \times \pi s^2 = \pi r s.$$

But what about if we don't know s, but only the height and the radius? Well you might notice that in our drawing of the cone we seem to have made a right-angled triangle, with the slanted side being the hypotenuse. We can therefore use Pythagoras' theorem to calculated the slant height:

$$s^2 = r^2 + h^2$$

So $s = \sqrt{r^2 + h^2}$. Therefore the area of our curved face is

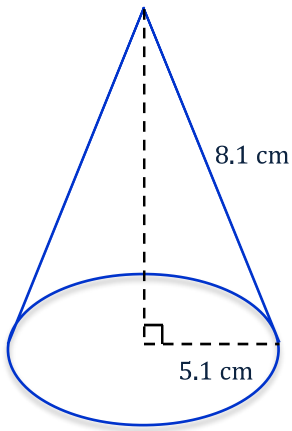
$$A = \pi r \sqrt{r^2 + h^2}.$$

So in summary the surface area of the two types of cones are (giving the formulas both in terms of r and s, and r and h:

Cone Type	Surface Area
Open	$A = \pi rs = \pi r\sqrt{r^2 + h^2}$
Closed	$A = \pi rs + \pi r^2 = \pi r\sqrt{r^2 + h^2} + \pi r^2$

Let's look at some examples.

Find the surface are of the following closed cone:

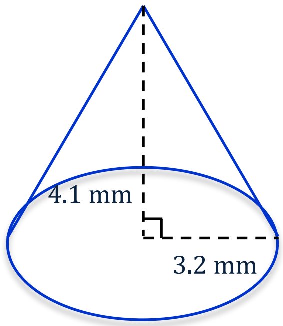


In this case we are given the slanted side length of 8.1 cm and the radius of 5.1 cm. Therefore the surface area of cone is:

$$\begin{aligned} A &= \pi \times 5.1 \text{ cm} \times 8.1 \text{ cm} + \pi \times 5.1^2 \text{ cm}^2 \\ &= 41.31\pi + 26.01\pi \text{ cm}^2 \\ &= 67.32\pi \text{ cm}^2 \\ &= 211.4920174 \ldots \text{ cm}^2 \end{aligned}$$

So the surface area is approximately 211.49 cm^2.

For next example, let's work out the surface are of the following open cone:



In this case we are given the height and radius so we use the formula $A = \pi r\sqrt{r^2 + h^2}$:

$$\begin{aligned} A &= \pi \times 3.2 \times \sqrt{3.2^2 + 4.1^2}\ \text{mm}^2 \\ &= \pi \times 3.2 \times \sqrt{27.05}\ \text{mm}^2 \\ &= 52.2857673 \ldots\ \text{mm}^2 \end{aligned}$$

So the surface area is approximately $52.29\ \text{mm}^2$.

If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

References

De Klerk, J. (2007). *Illustrated Maths Dictionary*. 4th ed. Pearson. Melbourne.