Volume and Surface Area

1. Volumes of Solids

INTRODUCTION

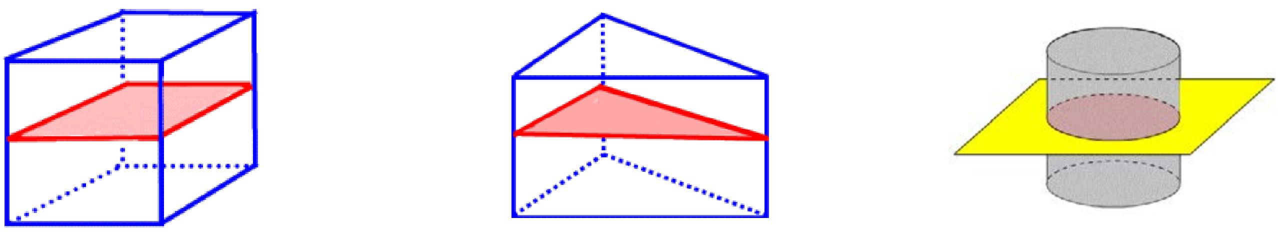
A *solid* is a three-dimensional (3D) object, that is, it has *length*, *width* (sometimes called *thickness* or *depth*) and *height*. It is this third dimension that distinguishes a solid from a *plane shape*. If you need to know more about plane shapes, areas and perimeters, please refer to *Area and Perimeter – 1. Areas of Plane Shapes* and *Area and Perimeter – 2. Perimeters of Plane Shapes*.

The words *volume* and *capacity* are often used interchangeably; however there is a subtle difference. "Volume is the amount of space occupied by an object or substance" whereas "capacity refers to the amount a container can hold … and generally refers to liquid measurement" (NSW Board of Studies, n. d.).

Volume is probably easiest to talk about if we think of a box. The boxes we are probably used to are rectangular in shape – in fact they are called rectangular prisms.

PRISMS AND CYLINDERS

A prism is a solid where the cross-section is always an identical polygon and the top and bottom faces are parallel. Here are some examples. The first is a rectangular prism, the second is a triangular prism, and the third is not a prism but a cylinder (it shares many similarities to a prism and is sometimes referred to, informally, as a "circular prism"). The cross-sections, top and bottom faces are rectangles, triangles and circles, respectively.



Diagrams retrieved January 24, 2013, from http://www.regentsprep.org/Regents/math/geometry/GG2/PrismPage.htm and http://hotmath.com/hotmath_help/topics/cross-sections.html

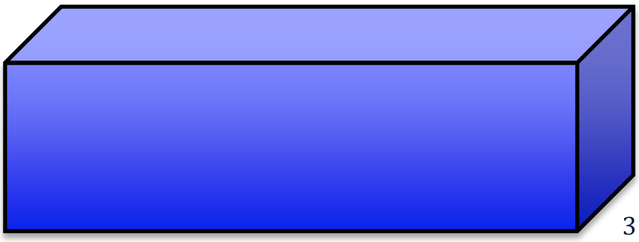
Think of the volume consisting of all the material in the container. Now try to imagine that all of that material could consist of layers of the material that have the same shape as the cross-section. Starting at the base, we would build up those layers, all the way to the top, which forms the last layer. If we know the area of the cross-section (which is the same as the area of the base) and we know how high the prism is, then we can multiply the area of the base by the height and the result is the volume.

In the rectangular prism the cross-section is a rectangle. We know that the area of a rectangle is $A = l \times w$ and is measure in square units (or units^2), where l is the length of the rectangle, and w is the width of the rectangle. If we let h be the height of the rectangular prism and V be its volume (which is measured in cubic units or units^3) then the volume of the rectangular prism is given by:

$$V = l \times w \times h$$

The volume is measure in cubic units because there are three dimensions to take into account. The units could be cm^3, mm^3, m^3 and so on.

Example: Calculate the volume of this rectangular container.



In this case we have $l = 12$ cm, $w = 3$ cm, and $h = 4$ cm. So:

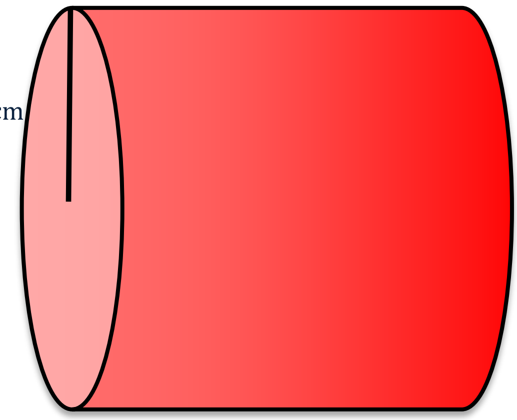
$$\begin{aligned} V &= 12\text{ cm} \times 3\text{ cm} \times 4\text{ cm} \\ &= 144\ cm^3 \end{aligned}$$

That is, the volume is $144\ cm^3$.

The volumes of other prisms/cylinders are calculated in the same way. Find the area of the base (which is the area of every cross-section) and multiply that by the height:

$$V = A \times h.$$

Sometimes the prism/cylinder will be on its side, so that its height will actually look like a length. The important thing is to make sure you label the straight direction as the height. For example, consider this cylinder:



The "base" of this prism is a circle because if you cut a cross-section vertically through the cylinder, you would always get a circle. So the volume is found by calculating the area of the circle and multiplying it by the height. For the circle:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times 6^2\text{ cm}^2 \\ &= 36\pi\text{ cm}^2 \\ &\approx 113.097\text{ cm}^2 \end{aligned}$$

Keep this result as exact as possible (that is, do not round it) by either remembering it as 36 times π or by storing the number in the calculator (remember the last result the calculator calculates is already stored in the *answer* key). So the volume is:

$$\begin{aligned} V &= 36\pi \times 10.3\text{ cm}^3 \\ &= 37.08\pi\text{ cm}^3 \\ &\approx 1164.903\text{ cm}^3 \end{aligned}$$

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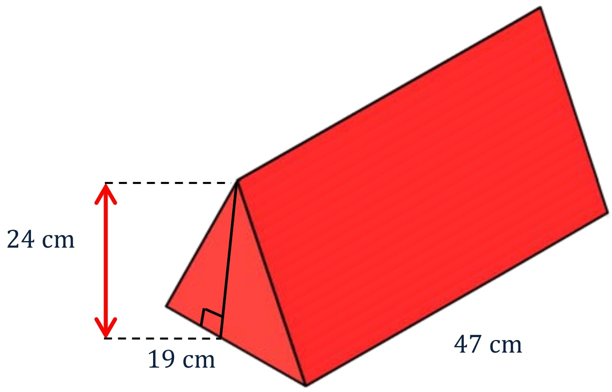
So the volume of the cylinder is (approximately) $1164.903\ \text{cm}^3$.

Here is an exercise for you to try. You can check your answer with the solutions at the end of the resource.

EXERCISES

1. Calculate the volume of the following figure.



PYRAMIDS/CONES

Another kind of solid you might be interested in is a pyramid.

Retrieved March 25, 2013, from http://www.flickr.com/photos/scottsm/8399190381/in/photostream

It can be shown that the volume of a pyramid is one-third the volume of its corresponding prism. "Corresponding" means that the pyramid and prism both have the same sized base and height, but the pyramid's side faces meet at a point, called the vertex. We can have a rectangular pyramid (the corresponding prism is a rectangular prism), a square pyramid (corresponding prism is a square prism), a triangular pyramid (corresponding prism is a triangular prism), a hexagonal pyramid…. When the base is a circle then we have a cone. In a similar way to how cylinder is not a prism, a cone is not a pyramid, but shares many of the same characteristics.

Since we know that the volume of a prism/cylinder is the area of the base times the height, this gives us a formula for the volume of a pyramid/cone:

$$V = \frac{1}{3}A \times h$$

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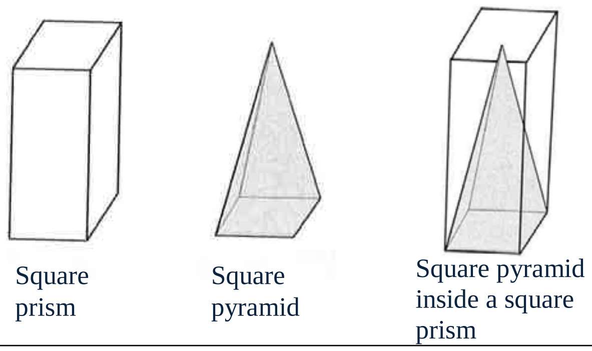
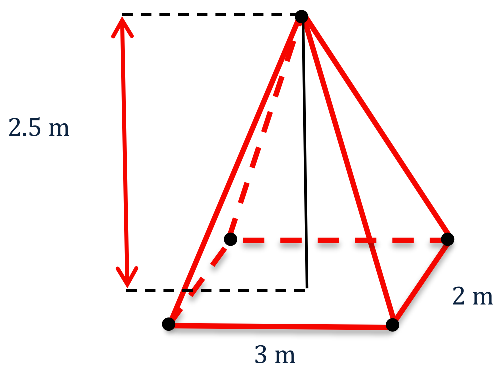
Where A is the area of the base of the pyramid/cone and h is the height of the pyramid/cone. When doing the calculation we must use the *perpendicular* height of the pyramid/cone, that is, it must be at right angles with the base of the pyramid/cone.

All pyramids have side faces that are triangles (the number of triangle faces is the number of sides of the base plane shape). Here we see a square pyramid and the corresponding square prism:



Diagrams retrieved January 30, 2013, from http://www.ck12.org/book/CK-12-Foundation-and-Leadership-Public-Schools%252C-College-Access-Reader%253A-Geometry/r1/section/7.6/

Let's consider an example. Calculate the volume of the following rectangular pyramid:



The area of the base of this pyramid is:

$$A = 3 \text{ m} \times 2 \text{ m} = 6 \text{ m}^2$$

Therefore the volume of the pyramid is:

$$V = \frac{1}{3} \times 6 \times 2.5 \text{ m}^3 = 5 \text{ m}^3$$

That is, the volume is 5 m^3.

Here is an exercise for you to try; you can check your result with the solution at the end of the resource.

EXERCISES

2. Calculate the volume of a cone where the radius of the circular base is 10.3 cm and its height is 20 cm. (You might like to draw a diagram, they are always useful.)

SPHERES

Spheres are a very different solid that only has one face, so there is no base like the other examples. To calculate the volume of a sphere you use the formula:

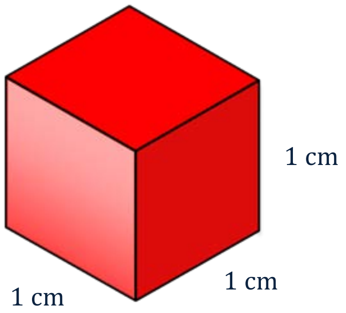
$$V = \frac{4}{3}\pi r^3.$$

CONVERTING BETWEEN CUBIC UNITS

When calculating volumes we need to make sure that all our variables are measured using the same units. That is, they all need to be in centimetres, or all in metres, etc. However, once we obtain our answer we may find that the units we chose are the best for the problem. For example, consider a rectangular prism that is 0.4 cm long, 0.2 cm wide, and 0.1 cm high. These units seem fine, but if we calculate the volume using them, we find that the volume is 0.008 cm^3. Such a small number in these units means there are probably better units to use.

We could start again and change all our lengths into millimetres, but it would be better if we could just change our units at this stage. Let's see how to change between cubic units.

Suppose we have a cube where all the dimension are 1 cm in length.



When we find the volume of the cube, we obtain:

$$\begin{aligned} V &= 1\text{ cm} \times 1\text{ cm} \times 1\text{ cm} \\ &= 1\text{ cm}^3 \end{aligned}$$

But we can also measure each dimension as 10 mm, and using this measure to calculate the volume, we obtain:

$$\begin{aligned} V &= 10\text{ mm} \times 10\text{ mm} \times 10\text{ mm} \\ &= 1000\text{ mm}^3 \end{aligned}$$

This means that

$$1\text{ cm}^3 = 1000\text{ mm}^3$$

So to move from cubic centimetres to cubic millimetres you multiply by 1000. In our example this means that our rectangular prism has a volume $V = 0.008 \times 1000\text{ mm}^3 = 8\text{ mm}^3$, which is much nicer to write. In the same way we also have that:

$$1\text{ m}^3 = 1000000\text{ cm}^3$$

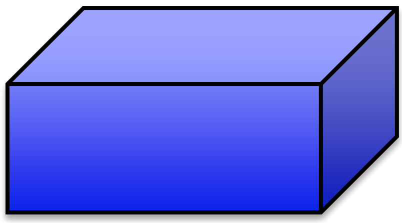
So to move from cubic metres to cubic centimetres you need to multiply by 1000000.

CONVERTING BETWEEN VOLUME UNITS AND CAPACITY UNITS

We have seen above that volume is measured in cubic units such as metres^3, centimetres^3 and so on. Capacity is measured in litres (L), millilitres (mL), kilolitres (kL) and so on. One millilitre is defined as the capacity of a one cubic centimetre volume, that is, **1 cm^3 holds 1 ml**. This means that one cubic metre folds 1000 litres, that is, **1 m^3 holds 1000 L**. Using the kilo prefix we could also write this as **1 m^3 holds 1 kL**

We can calculate the capacity of a solid, i.e the amount of liquid it might hold, by converting between units. For example, suppose a swimming pool is rectangular, has length 8 m, width 5 m, and can be filled to a height of 2 m. How many litres of water can it hold?

First, we draw the pool (not to scale):



The volume of the pool is

$$V = 8\text{ m} \times 5\text{ m} \times 2\text{ m}$$
$$= 80\text{ m}^3$$

Each m^3 holds 1000 L, and so the pool holds:

$$80 \times 1000 = 80000\text{ L} = 80\text{ kL}$$

Here is a couple for you to try. You can check your result with the solution at the end of the resource.

EXERCISES

3. Calculate the capacity of the cone in question 2.

4. Calculate the volume of a cylinder that has a radius of 10 cm and a height of 20 cm. How many litres can the cylinder hold? (It's a really good idea to draw a diagram to help you complete this sort of problem.)

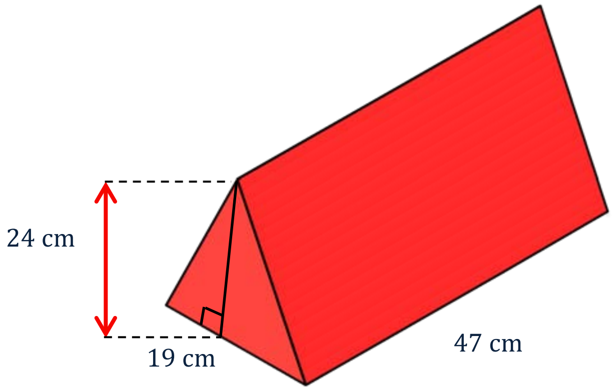
If you need help with any of the maths covered in this resource (or any other maths topic), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

References

NSW Board of Studies (n. d.). *NSW Syllabuses for the Australian Curriculum. Mathematics K-10 – Stage 2 – Measurement and Geometry: Volume and Capacity*. Retrieved February 18, 2013, from http://syllabus.bos.nsw.edu.au/mathematics/mathematics-k10/content/1127/

SOLUTIONS TO EXERCISES

1. Calculate the volume of the following figure.



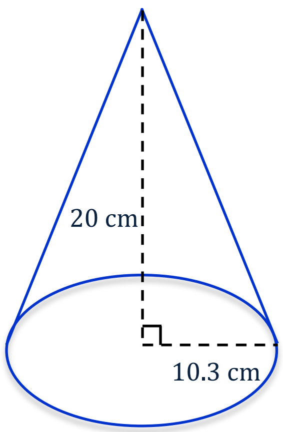
The “base” (cross section) of this prism is a triangle. The area of a triangle is $A = \frac{1}{2} \times b \times h$, where b is the length of the base and h is the height of the triangle. In our example the base (b) is 19 cm long and the height (h) is 24 cm. So the area of the triangle is:

$$A = \frac{1}{2} \times 19 \text{ cm} \times 24 \text{ cm} = 228 \text{ cm}^2.$$

Therefore the volume of the prism is:

$$V = 228 \text{ cm}^2 \times 47 \text{ cm} = 10716 \text{ cm}^3.$$

2. Calculate the volume of a cone where the radius of the circular base is 10.3 cm and its height is 20 cm. (You might like to draw a diagram, they are always useful.)
   A cone has a circular base, so our cone looks like:



   The formula for the volume of a cone is a third times the area of the base, times the height, so:

$$V = \frac{1}{3} \times \pi \times 10.3^2 \text{ cm}^2 \times 20 \text{ cm}$$
$$= 2221.943764 \ldots \text{cm}^3$$

   That is, the volume of the cone is approximately 2221.943 cm^3.
   You may have also done this problem by first working out the volume of the corresponding cylinder using the formula $V_{cyl} = \pi r^2 \times h$, then dividing by three.

3. Calculate the capacity of the cone in question 2.
   We found that the volume of the cone was approximately 2221.943 cm^3, since one cubic centimetre holds one millilitre, this means that the capacity of the cone is 2221.943 mL, or equivalently approximately 2.222 L.

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4. Calculate the volume of a cylinder that has a radius of 10 cm and a height of 20 cm. How many litres can the cylinder hold? (It's a really good idea to draw a diagram to help you complete this sort of problem.)
The area of the cylinder's cross section is given by
$$A = \pi r^2 = \pi \times 10^2 \text{ cm}^2 = 100\pi \text{ cm}^2$$
Therefore the volume of the cylinder is:
$$V = 100\pi \text{ cm}^2 \times 20 \text{ cm} = 2000\pi \text{ cm}^3 \approx 6283.185 \text{ cm}^3.$$
Converting this to millilitres we obtain that the capacity of the cylinder is approximately 6283.185 mL. To convert to litres we divide by 1000 to get 6.283185 L. We may like to round this to reduce the number of decimal places. To three decimal places our cylinder holds 6.283 L.