Algebra

11. Solving Quadratic Equations Using Algebra

Before looking at this resource, you might like to look at:

- *Algebra – 8. Factorising Quadratic Expressions*
- *Algebra – 9. Factorising More Complicated Quadratic Expressions*
- *Algebra – 10. Factorising Special Quadratic Expressions*

INTRODUCTION

An algebraic equation is different from an algebraic expression because the equation has an equals sign (“$=$”). An algebraic expression is simply a collection of algebraic terms, which may or may not be able to be simplified, whereas an algebraic equation must contain the equals sign, which separates two expressions, one of which may be as simple as 0. Quadratic expressions or equations always have 2 as the highest power of the variable (usually x), that is, there is always a term in x^2.

Examples of quadratic expressions:

- $x^2 + 5x - 2$
- $4x - 3x^2$
- $1 - x^2$
- $2x^2$

Examples of quadratic equations:

- $x^2 - 2x = 0$
- $5x^2 = 3$
- $2x + 1 = 4x^2$

Solving an equation means we are looking for the value(s) of the variable which makes the value of the left hand side of the equals sign the same as the value of the right hand side, that is, we are looking for the specific value(s) of x that make the equation true.

To solve a quadratic equation, we often factorise the quadratic expression involved. This means that we need to put all the non-zero terms together on one side of the equation and make it equal to zero. So, for the examples above, we would need:

- $x^2 - 2x = 0$ (already in the correct form)
- $5x^2 = 3 \rightarrow 5x^2 - 3 = 0$
- $2x + 1 = 4x^2 \rightarrow 4x^2 - 2x - 1 = 0$

Now we can either factorise the quadratic expression on the left hand side of the equation or we can use the quadratic formula.

SOLVING QUADRATIC EQUATIONS BY FACTORISING

Example 1. Solve $x^2 + 7x + 12 = 0$

Step 1: Put all the non-zero terms on the left hand side of the equals sign and 0 on the right. This example is already in this correct form.
Step 2: Factorise the quadratic expression on the left (see previous resources if you are not sure how to factorise).

$$x^2 + 7x + 12 = 0$$

$$(x + 4)(x + 3) = 0$$

Step 3: Now, on the left hand side of the equation, the two factors, $(x+4)$ and $(x+3)$, when multiplied together, make up the original quadratic expression. But we know that, when multiplied together, they also make 0 because that is the value on the right hand side. The only way that you can get 0 as the result of multiplying two numbers (or expressions) is that at least one of those numbers (or expressions) MUST be 0.

This means that we know, in our equation from Step 2, that either $x+4=0$ or $x+3=0$. If $x+4=0$, then $x=-4$, and if $x+3=0$, then $x=-3$.

So, we have solved our equation and the values that x can take are either -4 or -3.

Step 4: Check your answers. You can do this informally, either "in your head" or by using the calculator, or formally as follows, where "LHS" stands for the left hand side of the equals sign in the equation and "RHS" stands for the right hand side:

Check $x=-4$

$$\begin{aligned} LHS &= x^2+7x+12 \\ &= (-4)^2+7\times -4+12 \\ &= 16-28+12 \\ &= 0 = RHS \end{aligned}$$

$$\therefore x=-4$$

Check $x=-3$

$$\begin{aligned} LHS &= x^2+7x+12 \\ &= (-3)^2+7\times -3+12 \\ &= 9-21+12 \\ &= 0 = RHS \end{aligned}$$

$$\therefore x=-3$$

$\therefore$ the solution is $x=-4$ or -3

With the factorising method, it is very important to make sure that the factors are correct, so it is worthwhile expanding them out to make sure they equal the quadratic expression.

Example 2. Solve $12x^2+5x-2=0$

Again, the equation is in the correct form and is ready to factorise.

$$-24$$

$$12x^2+5x-2=0$$

$$\frac{(12x\quad)(12x\quad)}{12}=0$$

$$\frac{(12x+8)(12x-3)}{12}=0$$

$$(3x+2)(4x-1)=0$$

Now, we can see that one of our brackets must be 0, so we solve them separately:

$(3x+2)=0$	or	$(4x-1)=0$
$3x+2=0$		$4x-1=0$
$3x=-2$		$4x=1$
$x=-\frac{2}{3}$		$x=\frac{1}{4}$

Check the solutions:

Check $x=-\frac{2}{3}$

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$$LHS = 12\left(-\frac{2}{3}\right)^2 + 5\left(-\frac{2}{3}\right) - 2$$

$$= 12 \times \frac{4}{9} - \frac{10}{3} - 2$$

$$= \frac{48 - 30 - 18}{9}$$

$$= 0 = RHS$$

$$\therefore x = -\frac{2}{3}$$

Check $x = \frac{1}{4}$

$$LHS = 12\left(\frac{1}{4}\right)^2 + 5\left(\frac{1}{4}\right) - 2$$

$$= 12 \times \frac{1}{16} + \frac{5}{4} - 2$$

$$= \frac{3 + 5 - 8}{4}$$

$$= 0 = RHS$$

$$\therefore x = \frac{1}{4}$$

$\therefore$ the solution is $x = -\frac{2}{3}$ or $x = \frac{1}{4}$

Here are a couple for you to try. You can check them from the solutions at the back.

EXERCISES

Solve:

1. $x^2 - 3x + 2 = 0$

2. $2x^2 - x = 15$

SOLVING QUADRATIC EQUATIONS USING THE FORMULA

Step 1: Put all the non-zero terms on the left hand side of the equals sign and 0 on the right. You need to make sure that the terms are in the following order: firstly the term in x^2, next the term in x, and lastly the constant (the number on its own, without any x).

Step 2: Substitute the equation's coefficients into the quadratic formula:

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

What do a, b, and c stand for? Well, in Step 1, we put all our terms on the left hand side in a particular order. We have actually put them in the order $ax^2 + bx + c$, so a is the coefficient of (the number multiplied by) x^2, b is the coefficient of x and c is the constant term.

Step 3: Simplify the expression on the right hand side; you will need to be careful with the order of operations.

Step 4: Lastly, as above, check your answers.

We will solve the same examples as we did before, but this time we will use the quadratic formula.

Example 1. Solve $x^2 + 7x + 12 = 0$

List the values of a, b, and c

$$a = 1, b = 7, c = 12$$

Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain:

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4 \times 1 \times 12}}{2 \times 1}$$

Note that the $\pm$ tells us that we have two answers, one is found by adding the – 7 and the square root and then dividing that total by 2, and the other is found by subtracting the square root from – 7 and then dividing that total by 2.

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{(7)^2 - 4 \times 1 \times 12}}{2 \times 1} \\ &= \frac{-7 \pm \sqrt{49 - 48}}{2} \\ &= \frac{-7 \pm \sqrt{1}}{2} \end{aligned}$$

This is where we separate the answers:

$x = \frac{-7+1}{2}$ or $x = \frac{-7-1}{2}$

$= \frac{-6}{2}$ $= \frac{-8}{2}$

$= -3$ $= -4$

$\therefore$ the solution is $x = -4$ or -3, as before. (We have already checked these solutions so we will skip that step this time.)

Example 2. Solve $12x^2 + 5x - 2 = 0$

This is in the right form and we can just write down the values of a, b, and c:

$$a = 12, b = 5, c = -2$$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives:

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4 \times 12 \times -2}}{2 \times 12}$$

So,

$$x = \frac{-5 \pm \sqrt{25 + 96}}{24}$$

$$x = \frac{-5 \pm \sqrt{121}}{24}$$

$x = \frac{-5+11}{24}$ or $x = \frac{-5-11}{24}$

$= \frac{6}{24}$ $= -\frac{16}{24}$

So $x = \frac{1}{4}$ or $-\frac{2}{3}$, as before. (Again, we have already checked these solutions.)

Here is an example for which we must use the quadratic formula because it cannot be factorised easily.

Solve $3x^2 - x - 1 = 0$

It is in the correct form so we can just write down the values of a, b, and c:

$$a = 3,\ b = -1, \text{ and } c = -1$$

Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 3 \times -1}}{2 \times 3}$$

So,

$$x = \frac{1 \pm \sqrt{1 + 12}}{6}$$

and $x = \frac{1 \pm \sqrt{13}}{6}$.

We can leave our answers in that form (called the *exact value*), or, using a calculator, we can obtain a decimal approximation for each of the two answers, make sure you check what the question you are doing asks for. (In this case $x \approx 0.7676$ or $x \approx -0.4343$).

Here are a couple for you to try. You can check with the solutions at the back:

EXERCISES

Solve:

3. $5x^2 = 2x + 4$

4. $3x + 7 - 2x^2 = 0$

THE DISCRIMINANT, $b^2 - 4ac$

Notice how important the number under the square root sign is! In fact, it is so important that it has a special name: it is called *the discriminant*. We've just seen an example of when the discriminant does not have an exact square root and so the values of x are not rational, they are irrational, which, put very simply, means that they have a decimal value which goes on forever and doesn't repeat. Here is another case of the importance of the discriminant.

Solve $3x^2 = -2x - 5$

We first need to put this in the correct form, and we get:

$$3x^2 + 2x + 5 = 0$$

We cannot factorise, so we need to use the quadratic formula.

$$a = 3,\ b = 2,\ c = 5.$$

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times 5}}{2 \times 3} \\ &= \frac{-2 \pm \sqrt{4 - 60}}{6} \\ &= \frac{-2 \pm \sqrt{-56}}{6} \end{aligned}$$

Uh oh! We cannot find the square root of a negative number and so we are stuck! In this case, we say that x does not exist, and that the equation has *no real*[1] *roots.*

This means that the discriminant can tell us how many, and what sort of values x can take in a quadratic equation, which is the same as saying how many roots, and what sort of roots, the equation has.

1 *Real numbers* are any numbers you have met already. They can be positive, negative, whole numbers, fractions, and so on, even π. However, we cannot find any *real* value for the square root of a negative number – try it on your calculator! To take this square root we would need to venture into a whole different branch of mathematics *complex numbers*.

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Let's have a look at another example.

Solve $4x^2 - 4x + 1 = 0$

This time we could factorise, however we want to see what happens to the discriminant and so we'll use the formula. Later we'll also factorise to see what sort of expression the one on the left is (maybe you can do that now…).

$$a = 4, b = -4, c = 1.$$

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 4 \times 1}}{2 \times 4} \\ &= \frac{4 \pm \sqrt{16 - 16}}{8} \\ &= \frac{4 \pm \sqrt{0}}{8} \\ &= \frac{4}{8} = \frac{1}{2} \end{aligned}$$

Notice that this time the discriminant is 0, and that x has only one value. (Some people say that x has two equal values and the equation has two real, equal roots.)

We still need to check that $x = \frac{1}{2}$ is the correct solution to the equation.

$$\begin{aligned} LHS &= 4x^2 - 4x + 1 \\ &= 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 1 \\ &= 4 \times \frac{1}{4} - 2 + 1 \\ &= 1 - 2 + 1 = 0 = RHS \end{aligned}$$

And so $x = \frac{1}{2}$ is the correct solution.

Now, as mentioned, we could have factorised this equation to solve it and so here is that solution.

$$4x^2 - 4x + 1 = 0$$

$$\frac{(4x \quad)(4x \quad)}{4} = 0$$

$$\frac{(4x - 2)(4x - 2)}{4} = 0$$

$$(2x - 1)(2x - 1) = 0,$$

that is $(2x - 1)^2 = 0$ (a perfect square).

So each bracket is the same and we only find the same value of x from each, that is $x = \frac{1}{2}$.

Summing up the use of the discriminant, which is the value of $\boldsymbol{b^2 - 4ac}$ (note, it is the number UNDER the square root sign, not the square root of that number!):

- If $b^2 - 4ac > 0$, that is, if the number under the square root sign (the discriminant) is positive, then the quadratic equation will have 2 real, distinct roots (x will have 2 real values). In addition, if the discriminant, $b^2 - 4ac$ is positive AND it is also a perfect square, then the equation will have 2 real, distinct, rational roots (that is, they can be written as fractions).
- If $b^2 - 4ac = 0$, then the equation will only have 1 real root (or 2 equal real roots) and that root is rational.
- If $b^2 - 4ac < 0$, that is, if the discriminant is negative, then the quadratic equation will not have any real solution or roots (x will not have any values).

Using the discriminant, work out the number of solutions each of the following equations has (don't solve the equations):

EXERCISES

5. $8x^2 + 2x - 1 = 0$

6. $5x^2 - 3x + 7 = 0$

7. $4x^2 + 20x + 25 = 0$

If you need help with any of the maths covered in this resource (or any other maths topics), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

SOLUTIONS TO EXERCISES

1. $x^2 - 3x + 2 = 0$

   $(x - 2)(x - 1) = 0$

   $x = 2$, or $x = 1$

2. $2x^2 - x = 15$

   $2x^2 - x - 15 = 0$

   $\frac{(2x-6)(2x+5)}{2} = 0$

   $(x - 3)(2x + 5) = 0$

   $x = 3$, or $x = -\frac{5}{2}$

3. $5x^2 = 2x + 4$

   $5x^2 - 2x - 4 = 0$

   $a = 5, b = -2, c = -4$

   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

   $= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 5 \times -4}}{2 \times 5}$

   $= \frac{2 \pm \sqrt{4 + 80}}{10}$

   $= \frac{2 \pm \sqrt{84}}{10}$

   $= \frac{2 \pm 2\sqrt{21}}{10}$

   $\therefore x = \frac{1 \pm \sqrt{21}}{5}$

4. $3x + 7 - 2x^2 = 0$

   $2x^2 - 3x - 7 = 0$

   $a = 2, b = -3, c = -7$

   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

   $= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times -7}}{2 \times 2}$

   $= \frac{3 \pm \sqrt{9 + 56}}{4}$

   $\therefore x = \frac{3 \pm \sqrt{65}}{4}$

5. $8x^2 + 2x - 1 = 0$

   $b^2 - 4ac = 2^2 - 4 \times 8 \times -1$

   $= 4 + 32$

   $= 36$

   $\therefore$ the equation has two real rational roots.

6. $5x^2 - 3x + 7 = 0$

   $b^2 - 4ac = (-3)^2 - 4 \times 5 \times 7$

   $= 9 - 140$

   $= -131$

   $\therefore$ the equation has no real roots.

7. $4x^2 + 20x + 25 = 0$

   $b^2 - 4ac = 20^2 - 4 \times 4 \times 25$

   $= 400 - 400$

   $= 0$

   $\therefore$ the equation has one real rational root.