UNIVERSITY
OF WOLLONGONG
AUSTRALIA

Powers, Roots, and Logs

Logarithms

INTRODUCTION

Logarithms are used extensively in situations where numbers are unwieldy – either very large or very small. The main advantages of using logarithms (shortened to logs) in these cases are that the numbers become more comprehensible and calculations become much simpler.

One well-known use of logarithms is the Richter scale, used to measure the strength of earthquakes. Another is pH, a measure of acidity. Decibels (noise) is another example which uses logarithms as a unit of measure.

WHAT ARE THEY?

In *Powers, roots, and logs: Powers, indices, exponents* we mentioned that raising to a power is just shorthand for repeated multiplication, analogously to how multiplication is shorthand for repeated addition, i.e. how $4^3 = 4 \times 4 \times 4$ is similar to $3 \times 4 = 4 + 4 + 4$. We will now extend this analogy between powers and multiplication to show how taking a logarithm is similar to division.

From the equation $3 \times 4 = 12$ we can write an *equivalent equation* $12 \div 4 = 3$. The first equation asks what is 3 lots of 4 (and gives the answer 12), the second equation asks *how many* lots of 4 in 12 (and gives the answer 3). By writing $4^3 = 64$ we can ask what is 4 multiplied by itself 3 times (and answer 64), but how do we ask *how many* times do we need to multiply 4 by itself to get 64?

We use logarithms: $\log_4 64$, asks precisely this (so the answer is 3).

The equation $\log_4 64 = 3$ is an *equivalent equation* to $4^3 = 64$ (that is it gives the same information, just in a different way). We read the logarithm equation (or just *log* equation) as "the log[arithm] to *base* 4 of 64 is 3". Note that the number that is the base in the power equation, remains being called the base in the logarithm equation. Also note that the base of a logarithm is always a positive number, as will the number you take the log of.

Another example: $2^5 = 32$, so the equivalent logarithm equation is $\log_2 32 = 5$, because 5 is the power to which we raise (equivalently how many times we need to multiply) 2 to get 32. We read it as "log to base 2 of 32 is equal to 5" (or sometimes "log of 32 with base 2 is 5").

Further example: $3^{-1} = \frac{1}{3}$, so $\log_3 \left(\frac{1}{3}\right) = -1$ is the equivalent log equation.

EXERCISES

Write the equivalent log equations for the following:

1. $5^2 = 25$ 2. $10^2 = 100$ 3. $8^4 = 4096$ 4. $3^{-2} = \frac{1}{9}$

There is another way we could introduce logarithms in analogy to division. This is through *inverse operations* (if you don't know what these are you may like to consult *Solving and rearranging: Solving equations*). Division is the inverse operation of multiplication because it undoes a multiplication operation, i.e. $(3 \times 4) \div 4 = 3$, but how do we undo putting a number as a power?

That is what do we need to do to 4^3 to get the 3 back? We need to take the log: $\log_4(4^3) = 3$ (note $\log_4(4^3) = \log_4 64$). This works in general, just like how in general $(a \times b) \div b = a$:

$$\boxed{\log_b b^a = a}$$

Note: the inverse operation of division is multiplication so $(a \div b) \times b = a$, the same is true for logarithms i.e. its inverse is putting it as the power:

$$\boxed{b^{\log_b a} = a}$$

Now that we've mastered the relationship between powers and logs, the next trick is to work out parts of logarithm expressions, such as finding the correct base, number in the log, or the value of the logarithm. For example, you might be asked to find the value of: $\log_2 8$. So we need to work out how many times to multiply 2 by itself to get 8 (or equivalently work out the power to which 2 is raised to get 8).

Solution: we know $2 \times 2 \times 2 = 8$, so we need to multiply it by itself 3 times. Equivalently this means $2^3 = 8$ so that the equivalent equation is $\log_2 8 = 3$.

What about $\log_{10} 0.1$? We need to remember our place value, and that 0.1 is the same as $\frac{1}{10}$, which is the same as 10^{-1}. This means that $10^{-1} = 0.1$ so the equivalent equation is $\log_{10} 0.1 = -1$.

You can check any $\log_{10}$ on the calculator. To check $\log_{10} 0.1$ just press the button $\boxed{\text{log}}$ followed by 0.1 and the result should be -1. (Note that on the calculator, $\boxed{\text{log}}$ stands for "$\log_{10} \blacksquare$". You don't need to type the 10 into the calculator. Unfortunately, however, the only bases for logs on the calculator are 10 and the special number e, which you may work with later.)

EXERCISES

Find the value of the following terms:

5. $\log_2 4$	6. $\log_{10} 1000$	7. $\log_3 81$	8. $\log_{10} 0.01$
9. $\log_5 25$	10. $\log_5\left(\frac{1}{25}\right)$	11. $\log_2\left(\frac{1}{8}\right)$	12. $\log_{10}\left(\frac{1}{1000}\right)$

Suppose now that we have a log equation and we want to find the value we are taking the log of. Here is an example: $\log_2 x = 3$, what is x? We can write the equivalent equation using powers, i.e. $2^3 = x$, so $x = 8$.

EXERCISES

Find the value of x.

13. $\log_2 x = 5$	14. $\log_3 x = 4$	15. $\log_2 x = -3$	16. $\log_4 x = -1$

Now, suppose we have a log equation but we don't know the base. How do we find it? Well, let's call the base b. Here is an example: $\log_b 9 = 2$. We again need to write the equivalent equation that involves powers. In this case it would be $b^2 = 9$. So we are asking, what number when squared gives 9. So $b = \sqrt{9} = 3$. (Note that we don't have -3 as a solution, which also squares to give 9, because bases of logs are always positive!)

EXERCISES

Find the value of the base, b:

17. $\log_b 16 = 2$
18. $\log_b 81 = 4$
19. $\log_b 0.001 = -3$
20. $\log_b \left(\frac{1}{2}\right) = -1$

LOGS RESULTING IN FRACTIONS

So far every time we have taken a log, it has resulted in a whole number. This is not always (or even usually) the case. In this section we'll look at the case when the result will be a fraction, and how to find the result.

Example: find the value of $\log_4 8$.

We start as usual and think of it in terms of powers. What power to which 4 is raised will result in 8? You might try 2, but remember $4^2 = 4 \times 4 = 16$ (not $4 \times 2 = 8$). So it is not 2. And since 16 is larger than our required 8, the power should be less than 2. However, if we try 1 we only get $4^1 = 4$. So the correct number to raise 4 to is somewhere between 1 and 2.

What can we do? Well let's set x to be the value after taking the log, that is, let $x = \log_4 8$. Now we can write the equivalent equation in power form $4^x = 8$ (note this is precisely what we were doing with words in the previous paragraph).

Now for the trick: 4 is the same as 2^2, and 8 is the same as 2^3. So, we can write our power equation as $(2^2)^x = 2^3$! We can use our taking a power of a power rule to multiply the powers on the left hand side: $2^{2x} = 2^3$. Since we have the same base on both sides (2), this means that the powers must be also the same! That is $2x = 3$. This tells us $x = \frac{3}{2}$. So,

$$\log_4 8 = \frac{3}{2}.$$

(You can check this on your calculator by evaluating $4^{\frac{3}{2}}$ and making sure it gives 8.)

Here is an example using just the procedure: Find the value of $\log_9 243$.

Let $x = \log_9 243$, so $9^x = 243$. Therefore:

$$\begin{aligned} (3^2)^x &= 3^5 \\ 3^{2x} &= 3^5 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Therefore $\log_9 243 = \frac{5}{2}$.

Last of all, there are some rules which we can look at for logarithms as well.

LOG RULES

Remembering the relationship between logarithms and powers, this allows us to turn our power rules into logarithm rules. Our first rule for powers was $a^p \times a^q = a^{p+q}$, which says when you multiply two powers with the same base, you add the powers. Logs, as inverse of powers, do the opposite:

$$\boxed{\log_a x + \log_a y = \log_a(x \times y)}$$

That is, when you add two logs with the same base, you multiply their interiors. Similarly the rule $a^p \div a^q = a^{p-q}$, becomes:

$$\boxed{\log_a x - \log_a y = \log_a(x \div y)}$$

The third rule of powers was that when taking the power of a power, you multiply the powers. The log rule equivalent looks a little different:

$$\boxed{\log_a(x^p) = p \log_a(x)}$$

This says that if you take the log of a power, the power can come out the front and multiply the log. It is one of the most useful rules and allows us to solve for the power in an equation. Lastly we have a couple of special values when taking logs:

$$\boxed{\log_a 1 = 0} \quad \text{and} \quad \boxed{\log_a a = 1}$$

The last two rules can be proved using the equivalent equations technique starting from: $a^0 = 1$ and $a^1 = a$. The other three rules are a little harder to prove but can be found in many high school mathematics text books. We will only prove the first of the rules (the other two are similar):

Let $x = a^p$ and $y = a^q$, so that $p = \log_a x$ and $q = \log_a y$. We substitute these into $a^p \times a^q = a^{p+q}$ to obtain:

$$x \times y = a^{\log_a x + \log_a y}.$$

Writing this in its log equivalent for gives:

$$\log_a(x \times y) = \log_a x + \log_a y.$$

Which is what we wanted.

SOLUTIONS TO EXERCISES

1. $5^2 = 25$ becomes $\log_5 25 = 2$

2. $10^2 = 100$ becomes $\log_{10} 100 = 2$

3. $8^4 = 4096$ becomes $\log_8 4096 = 4$

4. $3^{-2} = \frac{1}{9}$ becomes $\log_3 \frac{1}{9} = -2$

5. $\log_2 4 = 2$ because $2^2 = 4$

6. $\log_{10} 1000 = 3$ because $10^3 = 1000$

7. $\log_3 81 = 4$ because $3^4 = 81$

8. $\log_{10} 0.01 = -2$ because $10^{-2} = \frac{1}{100} = 0.01$

9. $\log_5 25 = 2$ because $5^2 = 25$

10. $\log_5 \left(\frac{1}{25}\right) = -2$ because $5^{-2} = \frac{1}{25}$

11. $\log_2 \left(\frac{1}{8}\right) = -3$ because $2^{-3} = 8$

12. $\log_{10} \left(\frac{1}{1000}\right) = -3$ because $10^{-3} = \frac{1}{1000}$

13. $\log_2 x = 5$ becomes $2^5 = x$, so $x = 32$

14. $\log_3 x = 4$ becomes $3^4 = x$, so $x = 81$

15. $\log_2 x = -3$ becomes $2^{-3} = x$, so $x = \frac{1}{8}$

16. $\log_4 x = -1$ becomes $4^{-1} = x$, so $x = \frac{1}{4}$

17. $\log_b 16 = 2$ becomes $b^2 = 16$, so $b = \sqrt{16} = 4$

18. $\log_b 81 = 4$ becomes $b^4 = 81$, so $b = \sqrt[4]{81} = 3$

19. $\log_b 0.001 = -3$ becomes $b^{-3} = 0.001 = 10^{-3}$, so $b = 10$

20. $\log_b \left(\frac{1}{2}\right) = -1$ becomes $b^{-1} = \frac{1}{2} = 2^{-1}$, so $b = 2$