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PERFORMANCE OF THE 2SHI ESTIMATOR UNDER THE GENERALISED PITMAN NEARNESS CRITERION

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ABSTRACT

The paper considers an extension of Tran Van Hoa's family of 2SHI (two stage hierarchical information) estimators for the coefficient vector of a linear regression model and derives the conditions for the dominance of the 2SHI estimator over the OLS and Stein rule estimators under a Generalized Pitman Nearness (GPN) criterion when the disturbance variable is small.

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1. INTRODUCTION

In 1985, Tran Van Hoa proposed a family of 2SHI (two stage hierarchical information) estimators for the coefficient vector of the linear regression model. These 2SHI estimators were demonstrated to dominate in average mean squared errors (MSE) the OLS and the Stein estimators. A number of applications of the 2SHI estimators in empirical economic studies based on static and dynamic regression models where the 2SHI dominance was calculated have also been reported (see Tran Van Hoa 1992a, 1992b, 1993).

In 1990 and 1993, Tran Van Hoa and Chaturvedi extended the 2SHI further and considered a more general family of 2SHI estimators. They obtained the conditions for the dominance of the 2SHI estimator over the OLS and Stein rule estimators under a quadratic loss function. In those studies, the criterion of relative MSE or risks in the sense of Wald was adopted.

More recently, the Pitman nearness criterion has been used by several researchers for a comparison of alternative estimators, see Keating and Mason (1985), Rao et. al. (1986), Khatree (1987) and Peddada (1987), to cite a few. A special feature of this criterion is that it does not require the existence of the moments of the estimator and is less sensitive to the tail behaviour of the sampling distributions of the estimator. Rao et. al. (1986) and Keating & Mason (1988) considered a Generalized Pitman Nearness (GPN) criterion and analysed the performance of the Stein rule estimator in comparison to the Maximum Likelihood Estimator (MLE) for the mean of the multivariate normal distribution using extensive numerical studies. Sen et. al. (1989) derived the dominance condition for the Stein rule estimator over the MLE under a GPN criterion (see also Keating & Czitrom (1988) and Mason et. al. (1990)).

The main objective of the present paper is to establish the dominance of the 2SHI estimator over the OLS estimator and the Stein rule estimator under a GPN criterion. Since the technique adopted in Sen et. al. (1989) is quite involved and leads to fairly complicated expressions, we have adopted a simple methodology based on the small disturbances approximations.

2. THE MODEL AND THE ESTIMATORS

Consider the linear regression model

$$y = X\beta + \sigma u \qquad (1)$$

where y is a Tx1 vector of observations on the dependent variable, X is a Txk matrix of observations on k independent variables with full column rank, β is a kx1 vector of unknown regression coefficients and u is a Tx1 random vector following a multivariate normal distribution $N(0,I_T)$ independent of X and $\sigma^2(>0)$ is the disturbance variance.

In 1990 and 1993, Tran Van Hoa and Chaturvedi generalized the work by Tran Van Hoa (1985) and proposed the following family of explicit 2SHI estimators $\hat{\beta}_h$ for the coefficient vector β:

$$\hat{\beta}_h = \left[1 - cw\frac{(1-R^2)}{R^2} - c(1-w)\ \frac{R^2}{R^2\{1+c^*\,(1-R^2)/R^2\}}\right]b, \qquad (2)$$

where $b = (X'X)^{-1}X'y$ is the OLS estimator of β, $R^2 = (b'X'Xb/y'y)$ is the coefficient of determination corresponding to a no intercept model and w $(0\le w\le 1)$, $c(\ge 0)$ and c^* (≥ 0) are the characterizing scalars.

We can equivalently write the estimators $\hat{\beta}_h$ as:

$$\hat{\beta}_h = \left[1\text{-}cw\frac{(y\text{-}Xb)'(y\text{-}Xb)}{b'X'Xb} - c(1\text{-}w)\frac{(y\text{-}Xb)'(y\text{-}Xb)}{b'X'Xb + c^*(y\text{-}Xb)'(y\text{-}Xb)}\right] b, \tag{3}$$

It can be verified that, when $c^*=0$ or $w=1$, the 2SHI estimator $\hat{\beta}_h$ reduces to the following Stein rule estimator $\hat{\beta}_s$.

$$\hat{\beta}_s = \left[1 - \frac{(y\text{-}Xb)'(y\text{-}Xb)}{b'X'Xb}\right] b. \tag{4}$$

3. COMPARISON OF THE ESTIMATORS

For the comparison of the estimators, let us consider the quadratic loss function

$$M(\hat{\beta}) = (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta).$$

Then, following Rao et. al. (1986), the formal definition of the GPN criterion is given as follows:

DEFINITION:

For any two estimators $\hat{\beta}$ and $\tilde{\beta}$ of β, under the loss function above, the estimator $\hat{\beta}$ is said to be Pitman closer to the estimator $\tilde{\beta}$ if

$$P[M(\tilde{\beta}) - M(\hat{\beta}) > 0] > \frac{1}{2}.$$

THEOREM:

Under the assumption of small disturbances variance, up to order $0(\sigma)$, we have

$$P[M(b) - M(\hat{\beta}_h) > 0] = \frac{1}{2} + \frac{\sigma}{(2\pi\tau)^{1/2}} \; [(k\text{-}1) - \frac{c}{2}\,(T\text{-}k)]. \tag{5}$$

$$P[M(\hat{\beta}_s) - M(\hat{\beta}_h) > 0] = \frac{1}{2} - \frac{\sigma}{(2\pi\tau)^{1/2}} \; [(k\text{-}1) - c\,(T\text{-}k)]. \tag{6}$$

PROOF:

Let us define

$$z = (X'X)^{-1/2}\, X'u, \quad \delta = (X'X)^{-1/2}\,\beta, \quad ,\tau = \delta'\delta,$$

$$v = u'\,[I_T - X(X'X)^{-1}\, X'\,]\,u.$$

Then $z \sim N(0,I_k)$ and $v \sim \chi^2$ distribution with (T-k) degrees of freedom independently of z. Then, to order $0(\sigma^4)$, we have

$$(b-\beta)'X'X(b-\beta)-(\hat{\beta}_h - \beta)'X'X(\hat{\beta}_h - \beta)$$

$$= \; 2c\sigma^3\frac{v}{\tau}\,[\delta'z+\sigma\{z'z - \frac{2}{\tau}\,(\delta'z)^2 - \frac{cv}{2}\}\,],$$

which is greater than zero as long as

$$\eta = \frac{1}{\sqrt{\tau}}[\,\delta'z+\sigma\{z'z - \frac{2}{\tau}\,(\delta'z)^2 - \frac{cv}{2}\}\,],$$

is greater than zero. Let $f(\eta)$ denote the pdf of η. Then to order $0(\sigma)$

$$P[(M(b) - M(\hat{\beta}_h) > 0] \; = \int_0^{\infty} f(\eta)d\eta.$$

To obtain the pdf $f(\eta)$ we observe that, to order $0(\sigma)$, the characteristic function of η is given by:

$$\begin{aligned}
\psi(t) &= E[\exp(it\eta)] \\
&= E[\exp\{\frac{it}{\sqrt{\tau}}\left(\delta' z + \sigma z' z - \frac{2\sigma}{\tau}(\delta' z)^2 - \sigma\frac{cv}{2}\right)\}] \\
&= E[\exp\left(it\frac{\delta' z}{\sqrt{\tau}}\right)\{1 + \frac{it\sigma}{\sqrt{\tau}}\left(z'z - 2\frac{(\delta z)^2}{\tau} - \frac{cv}{2}\right)\}] \\
&= \exp(-\frac{1}{2}t^2)\,[1 + \frac{i\sigma}{\sqrt{\tau}}\{t(k-2) + t^3 - c\frac{(T-k)}{2}t\}]
\end{aligned}$$

Now, utilizing the inversion formula

$$f(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\eta}\,\psi(t)dt,$$

along with the results that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-it\eta - \frac{1}{2}t^2)\,dt = \phi(\eta),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t\exp(-it\eta - \frac{1}{2}t^2)\,dt = -i\eta\phi(\eta),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t^3\exp(-it\eta - \frac{1}{2}t^2)\,dt = -i(3\eta - \eta^3)\;\phi(\eta),$$

where $\phi(.)$ denotes the pdf of a standard normal variate, we get the following approximate expression for $f(\eta)$:

$$f(\eta) = \phi(\eta)\,[1 + \frac{\sigma}{\sqrt{\tau}}\{\eta(k+1) - \eta^3 - \frac{c}{2}\eta\,(T-k)\}\,].$$

Thus, to order $0(\sigma)$, we obtain

$$P[M(b) - M(\hat{\beta}_h) > 0]$$

$$= \int_0^{\infty} f(\eta)\, d\eta$$

$$= \frac{1}{2} + \frac{\sigma}{(2\pi\tau)^{1/2}} [(k-1) - \frac{c}{2} (T-k)],$$

which leads to (5).

Following Tran Van Hoa and Chaturvedi (1993), to order $0(\sigma^4)$, we can write

$$(\hat{\beta}_s-\beta)'X'X(\hat{\beta}_s-\beta)-(\hat{\beta}_h-\beta)'X'X(\hat{\beta}_h-\beta)$$

$$= 2cc^* (1-w)\sigma^5 \frac{v2}{\tau^2} [-\delta'z-\sigma\{z'z - \frac{4}{\tau} (\delta'z)^2 - cv\}],$$

which is greater than zero if and only if

$$\gamma = \frac{1}{\sqrt{\tau}} [-\delta'z+\sigma\{z'z - \frac{4}{\tau} (\delta'z)^2 - cv\}],$$

is less than zero.

Now, to order $0(\sigma)$, the characteristic function of γ is given by:

$$\psi^*(t) = \exp\left(-\frac{1}{2} t^2\right) \left[1 + i \frac{\sigma}{\sqrt{\tau}} \{ (k-4)\, t+3t^3 - cvt \} \right].$$

Hence, using the inversion formula, the pdf of γ, to order $0(\sigma)$, is given by

$$f^*(\gamma) = \phi(\gamma)\left[1 + \frac{\sigma}{\sqrt{\tau}}\{(k+5)\eta - 3\eta^3 - cv\eta\}\right].$$

Therefore, to order $0(\sigma)$, we have

$$P[M(\hat{\beta}_s) - M(\hat{\beta}_h) > 0] = P[\gamma<0]$$

$$= \int_{-\infty}^{0} f^*(\gamma)d\gamma$$

$$= \frac{1}{2} - \frac{\sigma}{\sqrt{2\pi\tau}}[(k-1) - c(T-k)],$$

which gives the result (6).

From the above theorem it follows that, up to the order of our approximations, under the GPN criterion, the 2SHI estimator $\hat{\beta}_h$ dominates the OLS estimator b whenever

$$0 < c < \frac{2(k-1)}{(T-k)}, k > 1; \qquad (7)$$

whereas, $\hat{\beta}_h$ dominates the Stein rule estimator $\hat{\beta}_s$ whenever

$$c > \frac{(k-1)}{(T-k)}, k>1; \qquad (8)$$

The dominance conditions (7) and (8) show that under the GPN criterion the 2SHI estimator dominates both the OLS estimator as well as the Stein rule estimator whenever c lies in interval [(k-1)/(T-k),2(k-1)/(T-k)] and k>1. Thus, if k>1, it is possible to improve upon the Stein rule estimator by using the 2SHI estimator.

4. SIMULATION RESULTS ON THE PERFORMANCE OF THE 2SHI

The performance, by means of a simple simulation study, of the 2SHI estimator over the OLS and Stein estimators in some linear regression models is given in the table below. The 27 different models used are characterized by different representative values of (a) the weight w, (b) the sample size T, and (c) the disturbance variance σ^2. In these models, w=(0.2, 0.5, and 0.8), T=(10,14 and 16), k=8, and $c=c^* = 1.5(k-1)/(T-k)$. The data for X are actual annual economic data obtained from the international 1995 DX database. The magnitude of σ^2 are defined over the range (0.01, 1, and 100). The simulation results are based on the averages from 100 statistical trials (larger numbers of trials have been tried but the findings remain basically unchanged). The dominance between say b and $\hat{\beta}_s$, denoted by $R(b/\hat{\beta}_s)$, is computed as $100[M(b)-M(\hat{\beta}_s)]/M(\hat{\beta}_s)$ where M(b) and $M(\hat{\beta}_s)$ are the average loss of b and $\hat{\beta}_s$ respectively. The calculation of the relative risk is similar for $R(b/\hat{\beta}_h)$ and $R(\hat{\beta}_s/\hat{\beta}_h)$.

From these simulation results, the 2SHI estimator dominates both the OLS and Stein estimators in all models. The smallness of σ as discussed above in the paper can have a fairly wide range of values, from 0.1 to 10 in our simulation study. The dominance does not seem to be greatly affected by these different values of σ.

TABLE I Performance of the 2SHI Estimators over the OLS and Stein: Simulation Results

Average R^2 (for all models) =									
	0.980	0.972	0.962	0.962	0.816	0.807	0.941	0.722	0.733
Weight w =	0.20								
T =		10			14			16	
s^2 =	0.01	1	100	0.01	1	100	0.01	1	100
$R(b / \hat{\beta}_s)$	4.51	4.85	6.97	3.85	18.71	19.80	4.80	20.42	19.02
$R(b / \hat{\beta}_h)$	22.18	23.48	25.01	20.80	95.67	115.74	26.41	122.95	119.49
$R(\hat{\beta}_s / \hat{\beta}_h)$	16.90	17.78	16.86	16.32	64.83	80.08	20.61	85.15	84.42
Weight w =	0.50								
T =		10			14			16	
s^2 =	0.01	1	100	0.01	1	100	0.01	1	100
$R(b / \hat{\beta}_s)$	11.39	11.95	12.87	9.97	48.07	53.14	12.53	56.24	53.97
$R(b / \hat{\beta}_h)$	21.91	22.92	24.74	20.64	92.94	112.42	26.16	119.14	116.84
$R(\hat{\beta}_s / \hat{\beta}_h)$	9.45	9.80	10.52	9.70	30.30	38.70	12.11	40.26	40.84
Weight w =	0.80								
T =		10			14			16	
s^2 =	0.01	1	100	0.01	1	100	0.01	1	100
$R(b / \hat{\beta}_s)$	18.24	19.22	20.73	16.51	70.10	84.81	20.93	84.72	91.80
$R(b / \hat{\beta}_h)$	22.25	23.54	27.98	20.82	97.85	116.87	26.44	119.83	121.25
$R(\hat{\beta}_s / \hat{\beta}_h)$	3.39	3.62	6.00	3.70	16.32	17.35	4.56	19.01	15.36

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