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On the cross-inefficiency approach based on the deviation variables framework

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Abstract

Ebrahimi, Dhamotharan, Ghasemi, and Charles (2022) in their study [A cross-inefficiency approach based on the deviation variables framework. Omega 111, 102668] proposed a set of epsilon-based benevolent and aggressive formulations for cross-inefficiency data envelopment analysis (DEA). They argue that their new method enhances the discriminatory power of DEA and possesses two distinguishing characteristics: guaranteeing *unique optimal weights* and *non-negative efficiency scores* in cross-inefficiency evaluations. However, upon our analysis of their method, it appears that these assertions do not appear to hold true in general. Specifically, we have identified that (i) their presented results cannot be reliably replicated, and the aggressive and benevolent models may experience infeasibility or unboundedness issues; (ii) there is a fundamental error in the proof of the main theorem supporting their approach and the benevolent and aggressive formulations may still yield non-unique optimal solutions.; and (iii) their generated efficiency scores may still be negative, despite their assertion of guaranteeing non-negativity.

Keywords: Data envelopment analysis; Discriminating power; Cross-inefficiency; Deviation variables; Multiple optimal solutions**.**

1. Introduction

In their recent study, Ebrahimi et al. (2022) proposed a set of epsilon-based secondary goal benevolent and aggressive formulations within the framework of cross-inefficiency data envelopment analysis (DEA) to tackle the issue of non-unique optimal solutions. However, upon attempting to replicate the results reported in Table 3 of Ebrahimi et al. (2022), we found that their results cannot be reproduced. We have provided the correct results in Table 1 of Appendix A in our paper. Our careful analysis revealed that their aggressive and benevolent models can be potentially infeasible or unbounded.

We have also identified a fundamental flaw in the proof of their main theoretical contribution, i.e., Theorem 2, which asserts that their aggressive and benevolent models always have a unique optimal solution. In Appendix B, we present two numerical counterexamples to support our mathematical reasoning, demonstrating that Theorem 2 is not true in general, and the aggressive/benevolent formulations cannot guarantee unique weights, (in)efficiency scores, and rankings.

The second major contribution of Ebrahimi et al. (2022) is the introduction of a new scheme to handle negative efficiency scores within the variable returns to scale (VRS) cross-inefficiency approach. Initially, we argue that negative efficiency scores can also exist in the constant returns to scale (CRS) cross-inefficiency evaluation if efficiency scores are calculated using the formula $1 - d_{pj}^*$, where d_{pj}^* represents the inefficiency of the jth decision making unit (DMU$_j$) with optimal weights derived from DMU$_p$. As a result, the presence of negative efficiency scores is not limited to the VRS scenario when efficiency scores are computed using the $1 - d_{pj}^*$ formula. Furthermore, we demonstrate that their proposed approach to address negative scores in the VRS case is not entirely effective and may still produce negative efficiency scores.

2. Cross-inefficiency approach and related secondary goals

Model (1) minimizes the deviation variable (inefficiency) to maximize the efficiency of the DMU under evaluation. Ebrahimi et al. (2022) used Model (1) for the self-evaluation of DMUs prior to conducting their cross-inefficiency (peer) evaluations:

$$
\begin{array}{lll}
\min & d_{pp} & \\
\text{s.t.} & \sum_{i=1}^{m} v_{ip}\, x_{ip} = 1, & \\
& \sum_{r=1}^{s} u_{rp}\, y_{rj} - \sum_{i=1}^{m} v_{ip}\, x_{ij} + d_{pj} - w_p = 0, & j = 1, \dots, n, \\
& u_{rp} \geq 0, & r = 1, \dots, s, \\
& v_{ip} \geq 0, & i = 1, \dots, m, \\
& d_{pj} \geq 0, & j = 1, \dots, n, \\
& w_p \; free\ in\ sign &
\end{array}
\tag{1}
$$

where the indices $j = 1, \dots, n,$ are for DMUs, $r,\ r = 1, \dots, s,$ are for outputs and $i,\ i = 1, \dots, m,$ are for inputs; x_{ij} is the value of the ith input for DMU$_j$, y_{rj} is the value of the rth output for DMU$_j$, v_{ip} is the weight of the ith input for DMU$_p$, u_{rp} is the weight of the rth output for DMU$_p$, w_p is a free variable that indicates the type of returns-to-scale prevalent at DMU$_p$, d_{pp} is the deviation variable (inefficiency) of DMU$_p$ using the best possible set of weights for DMU$_p$, and d_{pj} is the inefficiency of DMU$_j$ using optimal weights for DMU$_p$. When $w_p = 0$, the model corresponds to the CRS model.

Optimal weights derived from Model (1), i.e., v_{ip}^* and u_{rp}^*, may not be unique, resulting in non-unique cross-inefficiency scores and rankings (Mahdiloo et al., 2021). Similar to the cross-efficiency approach, a secondary goal can be employed to select a solution from multiple optima. However, using a secondary goal does not always result in unique optimal weights (Mahdiloo et al., 2021).

Ebrahimi et al. (2022) proposed Model (2) in order to obtain a unique optimal solution for Model (1).

$$\begin{aligned}
\min \quad & \delta_p = \textstyle\sum_{j=1, j\neq p}^{n} d_{pj} & & \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{m} v_{ip}\, x_{ip} = 1, & & \\
& \textstyle\sum_{r=1}^{s} u_{rp}\, y_{rp} - \sum_{i=1}^{m} v_{ip}\, x_{ip} + d_{pp}^{*} - w_p = 0, & & \\
& \textstyle\sum_{r=1}^{s} u_{rp}\, y_{rj} - \sum_{i=1}^{m} v_{ip}\, x_{ij} + d_{pj} - w_p = 0, & & j = 1, \dots, n,\ j \neq p, \\
& u_{rp} \geq \varepsilon, & & r = 1, \dots, s, \\
& v_{ip} \geq \varepsilon, & & i = 1, \dots, m, \\
& d_{pj} \geq 0, & & j = 1, \dots, n, \\
& w_p\ free\ in\ sign. & &
\end{aligned} \tag{2}$$

Model (2) applies the same guiding idea of the benevolent cross-efficiency model in Doyle & Green (1994) to select a solution from multiple optima. This model minimizes the inefficiencies of DMU$_j$, $j \neq p$, while keeping the inefficiency of DMU$_p$ unchanged. In addition to the benevolent model, Doyle & Green (1994) also proposed an aggressive model that minimizes the efficiency of DMU$_j$, $j \neq p$. Analogously, Ebrahimi et al. (2022) developed an aggressive cross-inefficiency approach by maximizing the sum of inefficiencies ($\sum_{j=1, j\neq p}^{n} d_{pj}$) in the objective function of Model (2).

Ebrahimi et al. (2022) state their main result as Theorem 2, asserting that Model (2) always has a unique optimal solution. Their proof of Theorem 2 relies on the following statement in Taha (2011, p. 116):

"When the objective function is parallel to a nonredundant **binding constraint** (i.e., a constraint that is satisfied as an equation at the optimal solution), the objective function can assume the same optimal value at more than one solution point, thus giving rise to alternative optima."

Invoking the above statement incorrectly, Ebrahimi et al. (2022) argue that

"However, it is noted that the coefficient vector of the objective function is not a multiplier of any coefficient vectors of the constraints set (either redundant or nonredundant). As a result, we conclude that the model has a unique optimal solution."

According to Taha (2011, p.116), condition C1: the objective function is parallel to a non-redundant binding constraint at optimal solutions is a sufficient condition for the existence of multiple optimal solutions for an LP problem. Considering the contrapositive of this proposition, the reverse condition of C1, — i.e., RC1: the objective function is not parallel to any non-redundant binding constraint at optimal solutions — is necessary for an LP problem to have a unique optimal solution. However, in their proof of Theorem 2, Ebrahimi et al. (2022) incorrectly infer that RC1 is sufficient for ensuring unique optima. To further elaborate that RC1 cannot be a sufficient criterion for unique optima, we consider the following simple LP problem, Model (3), with the unit cube in the first octant as the feasible set depicted in Figure 1. Obviously, Model (3) has multiple optimal solutions, all lying on the blue line segment connecting extreme optimal points E and F, yet the objective function is not parallel to any of the nonredundant binding constraints at these optimal solutions.

$$\begin{array}{ll} \max & z = x_1 + x_3 \\ \text{s.t.} & x_1 \leq 1, \\ & x_2 \leq 1, \\ & x_3 \leq 1, \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{3}$$

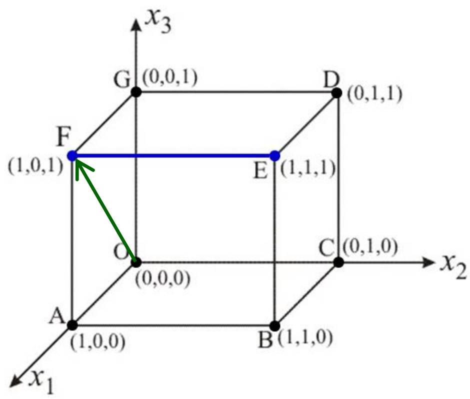

Figure 1. The occurrence of multiple optimal solutions where the objective function is not parallel to any of the binding constraints

To provide more corroborative evidence for supporting our mathematical reasoning that Theorem 2 is not true in general, we have presented two DEA-specific numerical counterexamples in Appendix B.

3. Negative efficiency scores

To address the issue of negative efficiency scores in the cross-inefficiency approach, Ebrahimi et al. (2022) proposed the following formula.

$$FCI_j = (1 - d_{jj}^*) \sum_{k=1}^{n} \frac{AI_{kj}}{n-1}, \qquad j = 1, \ldots, n, \tag{4}$$

where $AI_{kj} = 1 - AE_{kj}$ and $AE_{kj} = \frac{1-d_{kj}^*}{1-d_{jj}^*}$, $j = 1, \dots, n$; $k = 1, \dots, n$. In Ebrahimi et al's (2022) research, FCI_j is defined as the Farrell cross-inefficiency of DMU_j, AI_{kj} as the allocative inefficiency of DMU_j with respect to unit k ($k = 1, \dots, n$), AE_{kj} as the allocative efficiency of DMU_j with respect to unit k ($k = 1, \dots, n$), d_{jj}^* as the inefficiency score of DMU_j with its own (most favourable) weights, and d_{kj}^* as the inefficiency score of DMU_j with the optimal weights of other DMUs. Ebrahimi et al. (2022) asserted that $0 \leq FCI_j \leq 1$, and constructed the FCI_j formula based on the assumption that $0 \leq \frac{1-d_{kj}^*}{1-d_{jj}^*} \leq 1$. However, it should be noted that $\frac{1-d_{kj}^*}{1-d_{jj}^*}$ is not always less than or equal to 1, which may lead to negative values for AI_{kj} and FCI_j. Our calculations using the counterexample 1 dataset resulted in $FCI_1 = 3.1250$, $FCI_2 = 1.1667$ and $FCI_3 = -0.5521$, which all fall outside of the characterized range for FCI_j. Hence, it appears that Ebrahimi et al. (2022) did not effectively avoid negative efficiency scores in the cross-inefficiency approach.

Given that the deviation variables are set to be non-negative, a simple yet effective way to construct an efficiency score based on an optimal value of a deviation variable, i.e., d^*, is to define $e^* = \frac{1}{1+d^*}$. It is easy to see that $0 \leq e^* \leq 1$, since $d^* \geq 0$. A value of $e^* = 1$ indicates full efficiency, corresponding to a zero value of the deviation variable and the efficiency score decreases as the value of the deviation variable increases.

4. Unique versus non-unique optimal solutions: an operational test

In this section, we suggest an LP-model to check whether Model (2) has a unique optimal solution; and if that is not the case, the model generates an alternative optimal solution to Model (2). To construct such a test model, there are different approaches in the linear programming literature among which we follow the straightforward approach proposed by Appa (2002). Their approach is based on the following proposition.

Proposition (Appa, 2002). To check whether the *basic optimal solution*[1] $\mathbf{x}^*$ is a unique optimal solution to the general standard LP problem

[1] The notion of basic optimal solution is equivalent to *extreme optimal solution* in linear programming theory and is typically generated by a Simplex-based solver; for a more detailed account, see Murty (1983).

$$\begin{aligned} (\mathrm{P}) \quad \max \quad & \mathbf{cx} \\ \mathrm{s.t.} \quad & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

solve the following LP problem:

$$\begin{aligned} (\mathrm{P}') \quad \max \quad & \mathbf{ax} \\ \mathrm{s.t.} \quad & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{cx} = \mathbf{cx}^*, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

wherein $a_j = 1$ if $j \in J^0 := \{j : x_j^* = 0\}$ and $a_j = 0$ otherwise. Let the optimal solution to P' be denoted by $\mathbf{x}'^*$. Then, if $\mathbf{ax}'^* = 0$, $\mathbf{x}^* = \mathbf{x}'^*$ is the unique optimal solution to P, while if $\mathbf{ax}'^* > 0$, $\mathbf{x}^*$ and $\mathbf{x}'^*$ are two alternative solutions to P. We note that (P') might be unbounded that also indicates the existence of alternative optimal solutions.

To adopt the operational test P' to our case, we first transform Model (2) into an equivalent *standard LP* as in the form of P by (i) replacing the unrestricted variable w_p with the difference of two nonnegative variables, i.e., $w_p = w_p^1 - w_p^2,\ w_p^1 \geq 0,\ w_p^2 \geq 0$ and (ii) converting the lower bound constraints on weights—viz. $u_{rp} \geq \varepsilon,\ v_{ip} \geq \varepsilon, \forall\, i, r$—into equality constraints via introducing excess slack variables as $u_{rp} - t_{rp}^y = \varepsilon,\ v_{ip} - t_{ip}^x = \varepsilon,$ $t_{rp}^y \geq 0,\ t_{ip}^x \geq 0,\ \forall\, i, r$. Next, suppose that $(u_p^*, v_p^*, d_p^*, t_p^{x*}, t_p^{y*}, w_p^{1*}, w_p^{2*})$ is a basic optimal solution to the standard form of Model (2). Following Proposition 1, we constitute the objective vector of P' as $\mathbf{a}_\mathrm{p} = \left(\mathbf{a}_p^\mathrm{u}, \mathbf{a}_p^\mathrm{v}, \mathbf{a}_p^\mathrm{d}, \mathbf{a}_p^{tx}, \mathbf{a}_p^{ty}, a_p^{w1}, a_\mathrm{p}^{w2}\right)$ in which $a_{hp}^{\diamond} = 1$ $(\diamond \in \{u, v, d, tx, ty, w^1, w^2\})$ if the value of its corresponding variable in the optimal solution $(u_p^*, v_p^*, d_p^*, t_p^{x*}, t_p^{y*}, w_p^{1*}, w_p^{2*})$ is zero, and $a_{pj}^{\diamond} = 0$ otherwise, e.g., $a_{rp}^u = 1$ if $u_{rp}^* = 0$ and $a_{rp}^u = 0$ if $u_{rp}^* > 0$. Then, we solve the following LP problem as our operational test for optimality uniqueness of Model (2):

$$
\begin{aligned}
max \quad & \Omega_p = \boldsymbol{a}_p^u \boldsymbol{u}_p + \boldsymbol{a}_p^v \boldsymbol{v}_p + \boldsymbol{a}_p^d \boldsymbol{d}_p + \boldsymbol{a}_p^{tx} \boldsymbol{t}_p^x + \boldsymbol{a}_p^{ty} \boldsymbol{t}_p^y + a_p^{w1} w_p^1 + a_p^{w2} w_p^2 & & \\
s.t. \quad & \textstyle\sum_{i=1}^m v_{ip}\, x_{ip} = 1, & & \\
& \textstyle\sum_{r=1}^s u_{rp}\, y_{rp} - \sum_{i=1}^m v_{ip}\, x_{ip} + d_{pp}^* - (w_p^1 - w_p^2) = 0, & & \\
& \textstyle\sum_{r=1}^s u_{rp}\, y_{rj} - \sum_{i=1}^m v_{ip}\, x_{ij} + d_{pj} - (w_p^1 - w_p^2) = 0, & j = 1, \ldots, n,\ j \neq p, & \\
& u_{rp} - t_{rp}^y = \varepsilon, & r = 1, \ldots, s, & \quad (5) \\
& v_{ip} - t_{ip}^x = \varepsilon, & i = 1, \ldots, m, & \\
& \textstyle\sum_{j=1, j\neq p}^n d_{pj} = \delta_p^*, & & \\
& d_{pj}, t_{ip}^x, t_{rp}^y \geq 0, & \forall i, j, r, & \\
& w_p^1, w_p^2 \geq 0 & &
\end{aligned}
$$

where δ_p^* is the optimal objective of Model (2) when DMU_p is being evaluated. For the sake of more clarification, the objective function of Model (5), Ω_p, can be explicitly expressed as

$$\Omega_p = \textstyle\sum_{u_{rp}^*=0} u_{rp} + \sum_{v_{ip}^*=0} v_{ip} + \sum_{d_{pj}^*=0} d_{pj} + \sum_{t_{ip}^{x*}=0} t_{ip}^x + \sum_{t_{rp}^{y*}=0} t_{rp}^y + \sum_{w_p^{h*}=0} w_p^h$$

i.e., it is the sum of all variables that take zero value in the given basic optimal solution $(u_p^*, v_p^*, d_p^*, t_p^{x*}, t_p^{y*}, w_p^{1*}, w_p^{2*})$. Based on Proposition 1, we now conclude the following corollary:

Corollary 1. Model (2) has a unique optimal solution if and only if the optimal objective of Model (5) is zero, i.e., $\Omega_p^* = 0$. On the contrary, if $\Omega_p^* > 0$ (which covers the unbounded case of P′), then Model (5) generates an alternative solution to Model (2).

As an illustration, we tested our proposed Model (5) for DMU_2 in Counterexample 1. The optimal solutions reported in Table 3 are obtained using LINDO's What's*Best!* solver available as an Excel add-in. Alternatively, the CPLEX LP-solver of GAMS generates Solution 2 as an optimal solution for the standard form of Model (2) and by solving Model (5) based on this solution, Solution 1 is yielded as an alternative optimal solution with the positive objective value of $\Omega_p^* = 5.3333$.

Given that the occurrence of multiple optimal solutions plays a crucial role in the ranking results provided by a cross-efficiency/inefficiency evaluation, the operational test suggested in this section can be regarded a useful robustness checking tool; our suggested model can be easily modified and used in the existing DEA cross-efficiency procedures.

5. Concluding remarks

We have identified several significant issues with Ebrahimi et al.'s (2022) epsilon-based benevolent and aggressive formulations for cross-inefficiency data envelopment analysis, raising questions about the effectiveness of their approach. We provided a cautionary note regarding the utilization of this new approach, aimed at preventing potential errors in performance evaluation. Upon careful examination of Ebrahimi et al.'s (2022) method, we have found that their developed aggressive and benevolent models may experience infeasibility, unboundedness, and negative efficiency scores. Additionally, we have demonstrated that the aggressive and benevolent models can lead to multiple optimal solutions, which contradicts the formal statement of Theorem 2 in Ebrahimi et al.'s (2022) paper. A significant implication of this finding is to alert the occurrence of such multiple optimal solutions that can affect the robustness of ranking results.

Appendix A. Infeasible and unbounded solutions

Table 1 presents the results for both the aggressive and benevolent models under the CRS and VRS assumptions. The maximum feasible value of ε is calculated using a linear program as detailed in Appendix C of Ebrahimi et al. (2022). For the CRS model, the optimal value of ε is 0.000003, while for the VRS model, it is 0.010101. It should be noted that the value of ε for the VRS model in Ebrahimi et al. (2022) is incorrectly reported as 0.000020. Consequently, we computed the VRS models using both values of ε to address this discrepancy. The results indicate that for the majority of DMUs, both the aggressive and benevolent models are either infeasible or unbounded. As a result, it is not possible to replicate the FCI_j results reported in Ebrahimi et al. (2022).

It is worth noting that while Ebrahimi et al. (2022) stated on page 7 that they solved Model (1) with the additional constraints of $u_{rp} \geq \varepsilon$ and $v_{ip} \geq \varepsilon$, $\forall$ i, r, their results are actually generated applying the usual non-negativity constraints, $u_{rp} \geq 0$ and $v_{ip} \geq 0$, $\forall\, i, r$. This can be seen, for instance, in the VRS technical efficiency scores of DMUs UNC, Owen, and Georgetown, which are all reported as 1. Therefore, Ebrahimi et al.'s (2022) emphasis on page 7 that their "…developed models are epsilon-based. Hence, in contrast to the existing approaches, the input and output factors cannot be ignored in the performance evaluation process." is not correct.

Additionally, it is significant to highlight that adding $u_{rp} \geq \varepsilon$ and $v_{ip} \geq \varepsilon$ to Model (1) to tackle infeasible and unbounded solutions will not be effective. In fact, this will lead to a bigger issue: using the selected value for ε results in negative VRS technical efficiency scores for most DMUs. This can be observed, for instance, in the VRS efficiency scores of DMUs UNC, Owen, and Georgetown, which are -200.840, -305.483, and -258.218, respectively.

Table 1. Infeasible and unbounded solutions - benevolent and aggressive models

DMUs	CRS			VRS				
		ε =0.000003			ε = 0.010101		ε = 0.000020	
	d_{pp}^*	δ_p^* (Be.*)	δ_p^* (Ag.**)	d_{pp}^*	δ_p^* (Be.)	δ_p^* (Ag.)	δ_p^* (Be.)	δ_p^* (Ag.)
Stanford	0.480	Inf. ***	Inf.	0.266	Inf.	Inf.	Inf.	Inf.
Harvard	0.424	Inf.	Inf.	0.000	2977.030	Unb. ****	31.770	Unb.
MIT	0.336	Inf.	Inf.	0.046	Inf.	Inf.	Inf.	Inf.
Berkeley	0.239	Inf.	Inf.	0.037	Inf.	Inf.	Inf.	Inf.
Wharton	0.184	Inf.	Inf.	0.000	2974.370	Unb.	8.708	Unb.
Columbia	0.284	Inf.	Inf.	0.256	Inf.	Inf.	Inf.	Inf.
NYU	0.201	Inf.	Inf.	0.067	Inf.	Inf.	Inf.	Inf.
Chicago	0.168	Inf.	Inf.	0.000	5985.010	Unb.	7.324	Unb.
Tuck	0.184	Inf.	Inf.	0.000	Inf.	Inf.	5.312	158.100
UCLA	0.244	Inf.	Inf.	0.174	Inf.	Inf.	Inf.	Inf.
Kellogg	0.193	Inf.	Inf.	0.184	Inf.	Inf.	Inf.	Inf.
Foster	0.195	Inf.	Inf.	0.055	Inf.	Inf.	Inf.	Inf.
Darden	0.137	Inf.	Inf.	0.000	Inf.	Inf.	5.653	8.378
Duke	0.161	Inf.	Inf.	0.074	Inf.	Inf.	Inf.	Inf.
Yale	0.245	Inf.	Inf.	0.200	Inf.	Inf.	Inf.	Inf.
Olin	0.000	10.881	17.019	0.000	Inf.	Inf.	7.799	199.873
Cornell	0.111	Inf.	Inf.	0.108	Inf.	Inf.	Inf.	Inf.
Emory	0.133	Inf.	Inf.	0.000	Inf.	Inf.	Inf.	Inf.
Michigan	0.000	7.862	16.496	0.000	Inf.	Inf.	5.403	43.789
Texas	0.176	Inf.	Inf.	0.104	Inf.	Inf.	Inf.	Inf.
Tepper	0.105	Inf.	Inf.	0.063	Inf.	Inf.	Inf.	Inf.
Kelley	0.135	Inf.	Inf.	0.127	Inf.	Inf.	Inf.	Inf.
UNC	0.047	Inf.	Inf.	0.000	Inf.	Inf.	Inf.	Inf.
Owen	0.000	10.174	10.966	0.000	Inf.	Inf.	Inf.	Inf.
Georgetown	0.000	9.286	11.362	0.000	Inf.	Inf.	Inf.	Inf.

*Be.: Benevolant
**Ag.: Aggressive:
***Inf.: Infeasible
****Unb.: Unbounded

Appendix B. Illustrative counterexamples

To illustrate the non-unique optimal solutions of Model (2) more specifically in the DEA context, we present two numerical counterexamples below.

Counterexample 1. We start by analyzing the dataset in Table 2, which consists of three DMUs with a single input (*x*) and single output (*y*). Figure 2 depicts the VRS technology set, and its efficient frontier formed by these DMUs.

Table 2. Dataset of numerical Counterexample 1

DMU	*X*	*y*
1	1	1
2	3	4
3	12	7

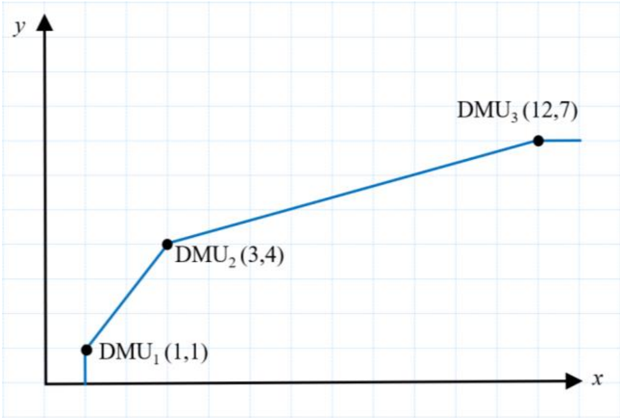

Figure 2. VRS technology and its efficient frontier

By applying Model (1), all three DMUs are identified as (strongly) BCC-efficient, i.e., $d_1^* = d_2^* = d_3^* = 0$. The maximum feasible value of ε in the VRS model is 0.0833. In Table 3, we report two profiles of optimal weights for the benevolent cross-inefficiency Model (2) when DMU$_2$ is being evaluated. From Table 3, it is evident that each profile of optimal weights results in different inefficiency scores and thus different rankings of DMUs, i.e., using $v^* = 0.3333$, $u^* = 1$, $w_p^* = 3$, DMU$_3$ is ranked higher than DMU$_1$ while with $v^* = 0.3333$, $u^* = 0.2222$, $w_p^* = -0.1111$, DMU$_1$ is ranked higher than DMU$_3$. This confirms the non-uniqueness of Model (2)'s optimal solutions and, therefore, disproves Theorem 2 in Ebrahimi et al. (2022) stating that Model (2) always has a unique optimal solution.

Table 3. Multiple optimal solutions for DMU 2 in Counterexample 1 and non-unique inefficiency scores (deviation variables).

Solutions	v^*	u^*	w_p^*	d_1^*	d_2^*	d_3^*	$\sum_{j=1}^{n} d_{pj}^*$
1	0.3333	1	3	2.3333	0	0	2.3333
2	0.3333	0.2222	-0.1111	0	0	2.3333	2.3333

Counterexample 2. Next, we look at the second example consisting of three DMUs, two inputs, and two outputs, with the dataset shown in Table 4.

Table 4. Dataset of Counterexample 2.

DMU	x_1	x_2	y_1	y_2
1	5	5	9	9
2	7	6	2	8
3	6	7	8	2

Using Model (1) with $w_p = 0$, the CRS inefficiency scores for DMUs 1, 2, and 3 are calculated as $d_1^* = 0$, $d_2^* = d_3^* = 0.5473$. The VRS inefficiency scores are $d_1^* = 0$, $d_2^* =$

$d_3^* = 0.8462$. The maximum feasible values of ε in the CRS and VRS models are 0.0427 and 0.0769, respectively. In Table 5, three profiles of optimal weights for DMU_1 based on the CRS Model (2), and, therefore, three sets of inefficiency scores are presented. As is evident from this table, rankings of DMU_2 and DMU_3 vary depending on the profiles of optimal weights, with DMU_3 being ranked higher than DMU_2 by the first two profiles, while DMU_2 is ranked higher than DMU_3 by the last profile.

Table 5. Multiple optimal solutions for DMU_1 and non-unique inefficiency scores – CRS case.

Solutions	v_1^*	v_2^*	u_1^*	u_2^*	d_1^*	d_2^*	d_3^*	$\sum_{j=1}^{n} d_{pj}^*$
1	0.1573	0.0427	0.0684	0.0427	0	0.8786	0.6103	1.4889
2	0.0427	0.1573	0.0684	0.0427	0	0.7641	0.7248	1.4889
3	0.1573	0.0427	0.0427	0.0684	0	0.7248	0.7641	1.4889

Similarly, the results of the VRS Model (2) are reported in Table 6, and once again, conflicting ranking results between DMU_2 and DMU_3 are apparent in this table.

Table 6. Multiple optimal solutions for DMU_1 and non-unique inefficiency scores – VRS case.

Solutions	v_1^*	v_2^*	u_1^*	u_2^*	w_p^*	d_1^*	d_2^*	d_3^*	$\sum_{j=1}^{n} d_{pj}^*$
1	0.1231	0.0769	0.0769	0.0769	0.3846	0	0.9385	0.8923	1.8308
2	0.0769	0.1231	0.0769	0.0769	0.3846	0	0.8923	0.9385	1.8308

It is worth noting that the VRS efficiency scores of both DMUs 2 and 3 are lower than their CRS scores, which is unusual and not anticipated. This abnormality is a result of using different epsilon values in the calculation of CRS and VRS scores. Therefore, when comparing these two scores —e.g., when calculating the scale efficiency of DMUs— it is important to exercise extra caution. This is a subtle aspect that has often been overlooked in existing literature.

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